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A solution to the thermo-elastic interface crack branching in dissimilar anisotropic bi-material media

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Abstract

An interface crack or delamination may often branch out of the interface in a laminated composite due to thermal stresses developing around the delamination/crack tip when the media is exposed to heat flow induced by environmental events such as a sudden short-duration fire. In this paper, the thermo-elastic problem of interface crack branching in dissimilar anisotropic bi-media is studied by using the theory of Stroh's dislocation formalism, extended to thermo-elasticity in matrix notation. Based on the complex variable method and the analytical continuation principle, the thermo-elastic interface crack/delamination problem is examined and a general solution in compact form is derived for dissimilar anisotropic bi-media. A set of Green's functions is proposed for the dislocations (conventional dislocation and thermal dislocation/heat vortex) in anisotropic bi-media. These functions may be more suitable than those which have appeared in the literature on addressing thermo-elastic interface crack branching in dissimilar anisotropic bi-materials. Using the contour integral method, a closed form solution to the interaction between the dislocations and the interface crack is obtained. Within the scope of linear fracture mechanics, the thermo-elastic problem of interface crack branching is then solved by modelling the branched portion as a continuous distribution of dislocations. The influence of thermal loading and thermal properties on the branching behavior is examined, and criteria for predicting interface crack branching are suggested, based on the extensive numerical results from the study of various cases.

Keywords: Dissimilar; Anisotropic; Composite; Interface; Delamination; Branching; Kinking; Fire; Thermal

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1. Introduction

Interface cracking may occur along the interface of two dissimilar media and could be one of the catastrophic failure modes for these materials. A common form of interface cracking is a delamination or a debond in laminate composite or sandwich structures. Williams (1959) employed an eigenfunction expansion method to study the stress distribution around the interface crack tip for a bi-material media consisting of two dissimilar isotropic infinite half planes and obtained a stress singularity in the form $r^{-\frac{1}{2}\pm i\epsilon}$, i.e. a solution of oscillating character. Since this pioneering work, many researchers contributed lots of effort and many useful studies have been published both for isotropic and anisotropic bi-material media. In particular, by using the Muskhelishvili's (1953) formalism, Erdogan (1963) obtained in 1963 a solution for several cracks aligned along the interface of a dissimilar isotropic bi-material media. England (1965) reconsidered this problem and quantitatively addressed the oscillatory character of an interface crack by focusing on the range of limits in which possible overlapping may occur. Rice and Sih (1965) studied in 1965 this problem by combining Muskhelishvili's (1953) complex-variable method with an eigenfunction expansion and formulated an expression for the stress intensity factors, as well as proposed a possible criterion for the interface crack growth. Suo and Hutchinson (1990) used in 1990 a dislocation distribution technique and supposition method to study a semi-infinite interface crack between the interface of two isotropic elastic layers. Extensive data were given in Suo and Hutchinson (1990) for practical application.

Clements (1971) started the investigation for dissimilar anisotropic bi-material media by using Stroh's sextic formalism (1958), then Willis (1971) using the Fourier transform method reconsidered this problem. Later on, Ting (1986) studied the asymptotic property of the interface crack in dissimilar anisotropic media by using an assumed stress function form and Qu and Li (1991) addressed this problem by applying the continuous interface dislocation distribution technique with real matrix notation.

It has been increasingly realized that the study of interface cracking can have significant practical interest due to the recently increasing use of laminated and sandwich composites in aerospace and marine structures, and the use of thin film structures in electronic packaging and computer components such as circuit board, etc. All these structures or devices often work in hostile environment where local temperature gradient fields are often experienced. A practical case of rapid built-up of thermal field gradients is when a loaded structure is exposed to fire on one side.

Studies on the influence of thermal loading on interface cracks can be traced from the 1960s. Several papers have been published on this subject such as Barber and Comninou (1982, 1983); Martin-Moran et al. (1983) and Chao and Shen (1993), etc.; these studies were, however, for isotropic bi-media; Atkinsion and Clements (1983) began to address the thermo-elastic interface crack problem for anisotropic bi-material media consisting of two dissimilar infinite half spaces. Later on, Hwu (1992) reconsidered the similar thermo-elastic interface crack problem in some details by employing the identities developed by Ting (1988). Choi and Thangjitham (1993) studied the interlaminar crack in laminated anisotropic composites by the Fourier integral transform technique; Herrmann and Loboda (2001) extended the Comninou (1977) contact model for interface cracks of dissimilar anisotropic bi-material media.

In contrast to the interface crack/delamination problems, the thermo-elastic interface crack branching problem in dissimilar bi-materials has received little attention. Our literature search revealed no analytical work on this problem. But, an interface delamination may easily branch out of the interface due to severe stress concentrations around the crack tip, especially the severe thermal stress concentrations when the structure is exposed to heat flow with or without mechanical loading. Therefore, the thermo-elastic interface crack branching phenomenon for dissimilar anisotropic bi-material media needs further investigations. The purpose of this paper is to analyze this phenomenon in terms of the dislocation theory (Eshelby et al., 1953).

The work presented in this paper is organized in the following way. In terms of the extended Stroh's (1958) anisotropic elasticity formulation (summarized in Appendix A), a general solution for a thermo-elastic interface delamination is first formulated by using the analytical continuation principle of complex functions. The procedure is similar to the one in Li and Kardomateas (2005).

Then, expressions for the thermal dislocation [thermal vortex, Dundurs and Comninou (1979)] and the conventional (or mechanical) dislocation, located in either of the bi-material components, are presented. To satisfy the continuation condition along the interface, a term accounting for the mixed thermal and mechanical interaction is introduced into these expressions. Then, a closed form solution is derived for the thermo-elastic interaction between the interface crack and the dislocation. Sub-sequentially, the branched crack is modelled by a continuous distribution of dislocations and a set of coupled singular integral equations in terms of the heat vortex density and the mechanical dislocation density is obtained. Subsequently, the strain energy release rate for the crack-kinked body is calculated and by maximizing it, the angle in favor of crack branching into one of the bi-material media can be found. Finally, several cases are numerically simulated to illustrate the thermal loading influence on the onset of interface crack branching and some important conclusions are drawn with regard to the criteria for the prediction of thermo-elastic crack/delamination branching in dissimilar anisotropic bi-material media.

2. A general solution to thermo-elastic interface crack in bi-media

The thermo-anisotropic elasticity in Stroh's formulation (1958) is summarized in Appendix A. In this section, the derivation of a general solution to the interface crack with thermal loading will be given by employing the complex variables method and the analytic continuity principle. A closed form solution to constant applied loading also will be given in this section.

2.1. A solution to the interface crack of anisotropic medium under thermo-mechanically combined loading

Let the medium I occupy the upper half space (denoted by L) and the medium II occupy the lower half space (denoted by R) (Fig. 1), then from Eq. (100) and (107) (Appendix A) one can have following expression for the bi-media:

$$
\begin{aligned}
\mathbf{u}^{\mathrm{I}} &= \mathbf{A}_{\mathrm{I}}\,\phi_{\mathrm{I}}(z_\alpha) + \overline{\mathbf{A}}_{\mathrm{I}}\,\overline{\phi_{\mathrm{I}}(z_\alpha)} + \mathbf{C}_{\mathrm{I}}\,\chi_{\mathrm{I}}(z_\tau) + \overline{\mathbf{C}}_{\mathrm{I}}\,\overline{\chi_{\mathrm{I}}(z_\tau)},\\
\varphi^{\mathrm{I}} &= \mathbf{B}_{\mathrm{I}}\,\phi_{\mathrm{I}}(z_\alpha) + \overline{\mathbf{B}}_{\mathrm{I}}\,\overline{\phi_{\mathrm{I}}(z_\alpha)} + \mathbf{D}_{\mathrm{I}}\,\chi_{\mathrm{I}}(z_\tau) + \overline{\mathbf{D}}_{\mathrm{I}}\,\overline{\chi_{\mathrm{I}}(z_\tau)},\\
T^{\mathrm{I}} &= \chi_{\mathrm{I}}'(z_\tau) + \overline{\chi_{\mathrm{I}}'(z_\tau)}, \quad h_2^{\mathrm{I}} = -\mathrm{i}k_{\mathrm{I}}\,\chi_{\mathrm{I}}''(z_\tau) + \mathrm{i}k_{\mathrm{I}}\,\overline{\chi_{\mathrm{I}}''(z_\tau)},
\end{aligned}
\tag{1}
$$

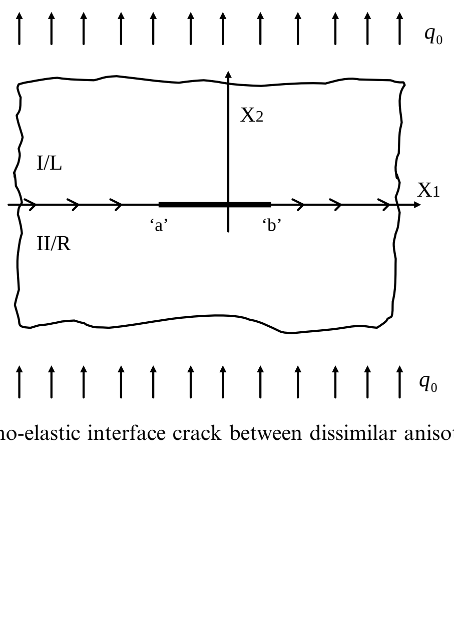

Fig. 1. A thermo-elastic interface crack between dissimilar anisotropic bi-media.

where $\mathbf{u}^{\mathrm{I}}$, φ^{I}, T^{I} are displacement, stress function and temperature fields for $z_\alpha \in L$

$$\begin{aligned}
\mathbf{u}^{\mathrm{II}} &= \mathbf{A}_{\mathrm{II}}\phi_{\mathrm{II}}(z_\alpha) + \overline{\mathbf{A}}_{\mathrm{II}}\overline{\phi_{\mathrm{II}}(z_\alpha)} + \mathbf{C}_{\mathrm{II}}\chi_{\mathrm{II}}(z_\tau) + \overline{\mathbf{C}}_{\mathrm{II}}\overline{\chi_{\mathrm{II}}(z_\tau)},\\
\varphi^{\mathrm{II}} &= \mathbf{B}_{\mathrm{II}}\phi_{\mathrm{II}}(z_\alpha) + \overline{\mathbf{B}}_{\mathrm{II}}\overline{\phi_{\mathrm{II}}(z_\alpha)} + \mathbf{D}_{\mathrm{II}}\chi_{\mathrm{II}}(z_\tau) + \overline{\mathbf{D}}_{\mathrm{II}}\overline{\chi_{\mathrm{II}}(z_\tau)},\\
T^{\mathrm{II}} &= \chi'_{\mathrm{II}}(z_\tau) + \overline{\chi'_{\mathrm{II}}(z_\tau)}, \quad h_2^{\mathrm{II}} = -\mathrm{i}k_{\mathrm{II}}\chi''_{\mathrm{II}}(z_\tau) + \mathrm{i}k_{\mathrm{II}}\overline{\chi''_{\mathrm{II}}(z_\tau)},
\end{aligned} \tag{2}$$

where $\mathbf{u}^{\mathrm{II}}$, φ^{II}, T^{II} are displacement, stress function and temperature fields for $z_\alpha \in R$.

For the convenience of writing, the symbols ‘I’ and ‘II’, denoting the quantities to medium ‘L’ and ‘R’, respectively, may be put as subscripts or subscripts. The interface crack is assumed to be located in the region $a < x_1 < b$, $-\infty < x_3 < \infty$ of the plane $x_2 = 0$. A heat flux h_0 and $\sigma_{i2}^{\infty} = p_i$ is applied at infinity (Fig. 1).

By the superposition principle and making use of Eq. $(106)_2$ in Appendix A, the boundary conditions for this problem can be written for the interface rack region ($a < x_1 < b$, $x_2 = 0$) as

$$\begin{aligned}
&h_{2+}^{\mathrm{I}}(x_1) = -h_0(x_1), \quad h_{2-}^{\mathrm{II}}(x_1) = -h_0(x_1),\\
&\varphi_+^{\prime\mathrm{I}}(x_1) = \varphi_-^{\prime\mathrm{II}}(x_1) = -p(x_1).
\end{aligned} \tag{3}$$

Along the interface outside the crack ($x_1 < a$ and $b < x_1$, $x_2 = 0$):

$$\begin{aligned}
&u_+^{\mathrm{I}}(x_1) = u_-^{\mathrm{II}}(x_1), \quad \varphi_+^{\prime\mathrm{I}}(x_1) = \varphi_-^{\prime\mathrm{II}}(x_1),\\
&T_+^{\mathrm{I}}(x_1) = T_-^{\mathrm{II}}(x_1), \quad h_{2+}^{\mathrm{I}}(x_1) = h_{2-}^{\mathrm{II}}(x_1);
\end{aligned} \tag{4}$$

and at infinity

$$h_2^{\mathrm{I}} = h_2^{\mathrm{II}} = 0, \quad \sigma_{ij}^{\mathrm{I}} = \sigma_{ij}^{\mathrm{II}} = 0, \tag{5}$$

where the convention $\phi(x_1, x_2) = \phi_\pm(x_1)$ as $x_2 \to 0^\pm$ for any function $\phi(x_1, x_2)$ was used and will be employed in the following sections.

The temperature continuity condition $(4)_3$ along the bonded interface gives

$$\begin{aligned}
&\chi'_{\mathrm{I}+}(x_1) + \overline{\chi}'_{\mathrm{I}-}(x_1) = \chi'_{\mathrm{II}-}(x_1) + \overline{\chi}'_{\mathrm{II}+}(x_1), \quad \text{or}\\
&\chi'_{\mathrm{I}+}(x_1) - \overline{\chi}'_{\mathrm{II}+}(x_1) = \chi'_{\mathrm{II}-}(x_1) - \overline{\chi}'_{\mathrm{I}-}(x_1).
\end{aligned} \tag{6}$$

One can define a function as

$$\theta(z) = \begin{cases} \chi'_{\mathrm{I}}(z) - \overline{\chi}'_{\mathrm{II}}(z), & z \in L,\\ \chi'_{\mathrm{II}}(z) - \overline{\chi}'_{\mathrm{I}}(z), & z \in R \end{cases} \tag{7}$$

which is analytical in the whole plane cut along the $a < x_1 < b$, then Eq. (6) is automatically satisfied. The heat flux continuity condition $(4)_4$ along the bonded interface gives

$$\begin{aligned}
&k_{\mathrm{I}}[\chi''_{\mathrm{I}+}(x_1) - \overline{\chi}''_{\mathrm{I}-}(x_1)] = k_{\mathrm{II}}[\chi''_{\mathrm{II}-}(x_1) - \overline{\chi}''_{\mathrm{II}+}(x_1)], \quad \text{or}\\
&k_{\mathrm{I}}\chi''_{\mathrm{I}+}(x_1) + k_{\mathrm{II}}\overline{\chi}''_{\mathrm{II}+}(x_1) = k_{\mathrm{II}}\chi''_{\mathrm{II}-}(x_1) + k_{\mathrm{I}}\overline{\chi}''_{\mathrm{I}-}(x_1).
\end{aligned} \tag{8}$$

Then a function can be defined as

$$\Theta(z) = \begin{cases} k_{\mathrm{I}}\chi''_{\mathrm{I}}(z) + k_{\mathrm{II}}\overline{\chi}''_{\mathrm{II}}(z), & z \in L,\\ k_{\mathrm{II}}\chi''_{\mathrm{II}}(z) + k_{\mathrm{I}}\overline{\chi}''_{\mathrm{I}}(z), & z \in R \end{cases} \tag{9}$$

which is analytical in the whole plane cut along the $a < x_1 < b$, Eq. (8) is automatically satisfied. Solving Eqs. (7) and (9) gives for $z \in L$:

$$\begin{aligned}
k_{\mathrm{I}}\chi''_{\mathrm{I}}(z) &= [k_{\mathrm{I}}\Theta(z) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'(z)]/[k_{\mathrm{I}} + k_{\mathrm{II}}],\\
k_{\mathrm{II}}\overline{\chi}''_{\mathrm{II}}(z) &= \Theta(z) - [k_{\mathrm{I}}\Theta(z) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'(z)]/[k_{\mathrm{I}} + k_{\mathrm{II}}];
\end{aligned} \tag{10}$$

and for $z \in R$:

$$
\begin{aligned}
k_{\mathrm{II}}\chi''_{\mathrm{II}}(z) &= [k_{\mathrm{II}}\Theta(z) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'(z)]/[k_{\mathrm{I}} + k_{\mathrm{II}}], \\
k_{\mathrm{I}}\overline{\chi}''_{\mathrm{II}}(z) &= \Theta(z) - [k_{\mathrm{II}}\Theta(z) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'(z)]/[k_{\mathrm{I}} + k_{\mathrm{II}}].
\end{aligned} \tag{11}
$$

Substituting Eq. (10) and (11) in condition $(3)_{1,2}$, one can obtain

$$
\begin{aligned}
&\frac{1}{k_{\mathrm{I}} + k_{\mathrm{II}}}[k_{\mathrm{I}}\Theta_+(x_1) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'_+(x_1)] - \Theta_-(x_1) + \frac{1}{k_{\mathrm{I}} + k_{\mathrm{II}}}[k_{\mathrm{I}}\Theta_-(x_1) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'_-(x_1)] = -\mathrm{i}h_0(x_1), \\
&\frac{1}{k_{\mathrm{I}} + k_{\mathrm{II}}}[k_{\mathrm{II}}\Theta_-(x_1) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'_-(x_1)] - \Theta_+(x_1) + \frac{1}{k_{\mathrm{I}} + k_{\mathrm{II}}}[k_{\mathrm{II}}\Theta_+(x_1) + k_{\mathrm{I}}k_{\mathrm{II}}\theta'_+(x_1)] = -\mathrm{i}h_0(x_1).
\end{aligned} \tag{12}
$$

Subtraction of Eq. $(12)_1$ from Eq. $(12)_1$ yields

$$
\Theta_+(x_1) - \Theta_-(x_1) = 0, \tag{13}
$$

which implies the function $\Theta(z)$ is also continuous along the region $a < x_1 < b$. Therefore this function is continuous along the whole interface.

By the statement of analytical continuation principle (Rudin, 1987), the function $\Theta(z)$ should be analytical on the whole plane. But by Liouville's theorem (Rudin, 1987), this function $\Theta(z)$ must be a constant function in the whole domain. However, the condition in Eq. $(5)_1$ imposes that this function should vanish at infinity. Therefore, this constant function must be identical to zero in the whole plane, i.e.

$$
\Theta(z) = 0, \quad \text{for all } z. \tag{14}
$$

Hence, following equations can be obtained from (9):

$$
\overline{\chi}''_{\mathrm{II}}(z) = -\frac{k_{\mathrm{I}}}{k_{\mathrm{II}}}\chi''_{\mathrm{I}}(z), \quad z \in L; \quad \overline{\chi}''_{\mathrm{I}}(z) = -\frac{k_{\mathrm{II}}}{k_{\mathrm{I}}}\chi''_{\mathrm{II}}(z), \quad z \in R. \tag{15}
$$

If the temperature field induced by the heat flux at the interface crack tends to zero at infinity, then integration of Eq. (15) gives:

$$
\overline{\chi}'_{\mathrm{II}}(z) = -\frac{k_{\mathrm{I}}}{k_{\mathrm{II}}}\chi'_{\mathrm{I}}(z), \quad z \in L; \quad \overline{\chi}'_{\mathrm{I}}(z) = -\frac{k_{\mathrm{II}}}{k_{\mathrm{I}}}\chi'_{\mathrm{II}}(z), \quad z \in R. \tag{16}
$$

Further integration of Eq. (16) leads to

$$
\overline{\chi}_{\mathrm{II}}(z) = -\frac{k_{\mathrm{I}}}{k_{\mathrm{II}}}\chi_{\mathrm{I}}(z), \quad z \in L; \quad \overline{\chi}_{\mathrm{I}}(z) = -\frac{k_{\mathrm{II}}}{k_{\mathrm{I}}}\chi_{\mathrm{II}}(z), \quad z \in R, \tag{17}
$$

where a constant contributing to rigid body motion is dropped. Eq. (7) turns to

$$
\theta(z) = \begin{cases} \left[1 + \frac{k_{\mathrm{I}}}{k_{\mathrm{II}}}\right]\chi'_{\mathrm{I}}(z), & z \in L, \\ \left[1 + \frac{k_{\mathrm{II}}}{k_{\mathrm{I}}}\right]\chi'_{\mathrm{II}}(z), & z \in R. \end{cases} \tag{18}
$$

Then both Eq. $(12)_1$ and $(12)_2$ become

$$
\theta'_+(x_1) + \theta'_-(x_1) = -\frac{k_{\mathrm{I}} + k_{\mathrm{II}}}{k_{\mathrm{I}}k_{\mathrm{II}}}\mathrm{i}h_0(x_1), \quad a < x_1 < b. \tag{19}
$$

The displacement continuity along the bonded interface gives

$$
A_{\mathrm{I}}\phi_{\mathrm{I}+}(x_1) + \overline{A}_{\mathrm{I}}\overline{\phi}_{\mathrm{I}-}(x_1) + C_{\mathrm{I}}\chi_{\mathrm{I}+}(x_1) + \overline{C}_{\mathrm{I}}\overline{\chi}_{\mathrm{I}-}(x_1) = A_{\mathrm{II}}\phi_{\mathrm{II}-}(x_1) + \overline{A}_{\mathrm{II}}\overline{\phi}_{\mathrm{I}+}(x_1) + C_{\mathrm{II}}\chi_{\mathrm{II}-}(x_1) + \overline{C}_{\mathrm{II}}\overline{\chi}_{\mathrm{II}+}(x_1)
$$

or

$$
\begin{aligned}
&A_{\mathrm{I}}\phi_{\mathrm{I}+}(x_1)-\overline{A}_{\mathrm{II}}\overline{\phi}_{\mathrm{II}+}(x_1)+C_{\mathrm{I}}\chi_{\mathrm{I}+}(x_1)-\overline{C}_{\mathrm{II}}\overline{\chi}_{\mathrm{II}+}(x_1)\\
&\quad=A_{\mathrm{II}}\phi_{\mathrm{II}-}(x_1)-\overline{A}_{\mathrm{I}}\overline{\phi}_{\mathrm{I}-}(x_1)+C_{\mathrm{II}}\chi_{\mathrm{II}-}(x_1)-\overline{C}_{\mathrm{I}}\overline{\chi}_{\mathrm{I}-}(x_1).
\end{aligned}\tag{20}
$$

Define the function

$$
\Phi(z)=\begin{cases}A_{\mathrm{I}}\phi_{\mathrm{I}}(z)-\overline{A}_{\mathrm{II}}\overline{\phi}_{\mathrm{II}}(z)+C_{\mathrm{I}}\chi_{\mathrm{I}}(z)-\overline{C}_{\mathrm{II}}\overline{\chi}_{\mathrm{II}}(z), & z\in L,\\ A_{\mathrm{II}}\phi_{\mathrm{II}}(z)-\overline{A}_{\mathrm{I}}\overline{\phi}_{\mathrm{I}}(z)+C_{\mathrm{II}}\chi_{\mathrm{II}}(z)-\overline{C}_{\mathrm{I}}\overline{\chi}_{\mathrm{I}}(z), & z\in R\end{cases}\tag{21}
$$

or

$$
\Phi(z)=\begin{cases}A_{\mathrm{I}}\phi_{\mathrm{I}}(z)-\overline{A}_{\mathrm{II}}\overline{\phi}_{\mathrm{II}}(z)+[k_{\mathrm{II}}C_{\mathrm{I}}+k_{\mathrm{I}}\overline{C}_{\mathrm{II}}]\chi_{\mathrm{I}}(z)/k_{\mathrm{II}}, & z\in L,\\ A_{\mathrm{II}}\phi_{\mathrm{II}}(z)-\overline{A}_{\mathrm{I}}\overline{\phi}_{\mathrm{I}}(z)+[k_{\mathrm{I}}C_{\mathrm{II}}+k_{\mathrm{II}}\overline{C}_{\mathrm{I}}]\chi_{\mathrm{II}}(z)/k_{\mathrm{I}}, & z\in R,\end{cases}\tag{22}
$$

where Eq. (16) was used.Differentiation of Eq. (22) and making use of (18) yields

$$
\Phi'(z)=\begin{cases}A_{\mathrm{I}}\phi'_{\mathrm{I}}(z)-\overline{A}_{\mathrm{II}}\overline{\phi}'_{\mathrm{II}}(z)+e_1\theta(z), & z\in L,\\ A_{\mathrm{II}}\phi'_{\mathrm{II}}(z)-\overline{A}_{\mathrm{I}}\overline{\phi}'_{\mathrm{I}}(z)+\overline{e}_1\theta(z), & z\in R,\end{cases}\tag{23}
$$

where $e_1=[k_{\mathrm{II}}C_{\mathrm{I}}+k_{\mathrm{I}}\overline{C}_{\mathrm{II}}]/[k_{\mathrm{I}}+k_{\mathrm{II}}]$ is a constant vector. Similarly, stress continuity on the bonded interface leads to:

$$
B_{\mathrm{I}}\phi'_{\mathrm{I}+}(x_1)+\overline{B}_{\mathrm{I}}\overline{\phi}'_{\mathrm{I}-}(x_1)+D_{\mathrm{I}}\chi'_{\mathrm{I}+}(x_1)+\overline{D}_{\mathrm{I}}\overline{\chi}'_{\mathrm{I}-}(x_1)=B_{\mathrm{II}}\phi'_{\mathrm{II}-}(x_1)+\overline{B}_{\mathrm{II}}\overline{\phi}'_{\mathrm{II}+}(x_1)+D_{\mathrm{II}}\chi'_{\mathrm{II}-}(x_1)+\overline{D}_{\mathrm{II}}\overline{\chi}'_{\mathrm{II}+}(x_1)
$$

or

$$
\begin{aligned}
&B_{\mathrm{I}}\phi'_{\mathrm{I}+}(x_1)-\overline{B}_{\mathrm{II}}\overline{\phi}'_{\mathrm{II}+}(x_1)+D_{\mathrm{I}}\chi'_{\mathrm{I}+}(x_1)-\overline{D}_{\mathrm{II}}\overline{\chi}'_{\mathrm{II}+}(x_1)\\
&\quad=B_{\mathrm{II}}\phi'_{\mathrm{II}-}(x_1)-\overline{B}_{\mathrm{I}}\overline{\phi}'_{\mathrm{I}-}(x_1)+D_{\mathrm{II}}\chi'_{\mathrm{II}-}(x_1)-\overline{D}_{\mathrm{I}}\overline{\chi}'_{\mathrm{I}-}(x_1).
\end{aligned}\tag{24}
$$

A function which automatically satisfies the condition (24) can be defined as:

$$
\omega(z)=\begin{cases}B_{\mathrm{I}}\phi'_{\mathrm{I}}(z)-\overline{B}_{\mathrm{II}}\overline{\phi}'_{\mathrm{II}}(z)+e_2\theta(z), & z\in L,\\ B_{\mathrm{II}}\phi'_{\mathrm{II}}(z)-\overline{B}_{\mathrm{I}}\overline{\phi}'_{\mathrm{I}}(z)+\overline{e}_2\theta(z), & z\in R.\end{cases}\tag{25}
$$

This function is analytical on the whole plane except the cut along the interface crack and in which $e_2=[k_{\mathrm{II}}D_{\mathrm{I}}+k_{\mathrm{I}}\overline{D}_{\mathrm{II}}]/[k_{\mathrm{I}}+k_{\mathrm{II}}]$ is a constant vector. From Eq. (22) and (25), one can obtain

$$
\begin{aligned}
&B_{\mathrm{I}}\phi'_{\mathrm{I}}(z)=\mathrm{i}N[\Phi'(z)-e_1\theta(z)]+N\overline{M}_{\mathrm{II}}^{-1}[\omega(z)-e_2\theta(z)],\\
&\overline{B}_{\mathrm{II}}\overline{\phi}'_{\mathrm{II}}(z)=B_{\mathrm{I}}\phi'_{\mathrm{I}}(z)-\omega(z)+e_2\theta(z)
\end{aligned}\tag{26}
$$

for $z\in L$;

$$
\begin{aligned}
&B_{\mathrm{II}}\phi'_{\mathrm{II}}(z)=\mathrm{i}\overline{N}[\Phi'(z)-\overline{e}_1\theta(z)]+\overline{N}\,\overline{M}_{\mathrm{I}}^{-1}[\omega(z)-\overline{e}_2\theta(z)],\\
&\overline{B}_{\mathrm{I}}\overline{\phi}'_{\mathrm{I}}(z)=B_{\mathrm{II}}\phi'_{\mathrm{II}}(z)-\omega(z)+\overline{e}_2\theta(z)
\end{aligned}\tag{27}
$$

for $z\in R$. Substituting Eq. (26) and (27) into the condition $(3)_{3,4}$, respectively, gives:

$$
\begin{aligned}
&B_{\mathrm{I}}\phi'_{\mathrm{I}+}(x_1)+B_{\mathrm{II}}\phi'_{\mathrm{II}-}(x_1)-\omega_-(x_1)+\overline{e}_2\theta_-(x_1)+\frac{k_{\mathrm{II}}}{k_{\mathrm{I}}+k_{\mathrm{II}}}[D_{\mathrm{I}}\theta_+(x_1)-\overline{D}_{\mathrm{I}}\theta_-(x_1)]=-p(x_1),\\
&B_{\mathrm{II}}\phi'_{\mathrm{II}-}(x_1)+B_{\mathrm{I}}\phi'_{\mathrm{I}+}(x_1)-\omega_+(x_1)+e_2\theta_+(x_1)+\frac{k_{\mathrm{I}}}{k_{\mathrm{I}}+k_{\mathrm{II}}}[D_{\mathrm{II}}\theta_-(x_1)-\overline{D}_{\mathrm{II}}\theta_+(x_1)]=-p(x_1),
\end{aligned}\tag{28}
$$

where Eqs. (16) and (18) are used.

Subtraction of Eq. $(28)_2$ from $(28)_1$ yields

$$\omega_+(x_1) - \omega_-(x_1) = 0 \tag{29}$$

which means that the $\omega(z)$ is continuous on the whole interface. By a similar argument as the one used in obtaining Eq. (14), one can conclude that

$$\omega(z) = 0, \quad \text{for all } z. \tag{30}$$

Either Eq. $(28)_1$ or $(28)_2$ leads to

$$\Phi'_+(x_1) + N^{-1}\overline{N}\Phi'_-(x_1) = \mathrm{i}N^{-1}[p(x_1) + \varrho_1\theta_+(x_1) + \varrho_2\theta_-(x_1)], \quad a \leqslant x_1 \leqslant b, \tag{31}$$

where

$$\varrho_1 = \frac{k_{\mathrm{II}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}D_I - N[\mathrm{i}e_1 + \overline{M}_{\mathrm{II}}^{-1}e_2], \quad \varrho_2 = \frac{k_{\mathrm{I}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}D_{\mathrm{II}} - \overline{N}[\mathrm{i}\overline{e}_1 + \overline{M}_{\mathrm{I}}^{-1}\overline{e}_2]; \quad N^{-1} = M_{\mathrm{I}}^{-1} + \overline{M}_{\mathrm{II}}^{-1}. \tag{32}$$

The general solutions to Eqs. (19) and (31) can be obtained by employing the contour integral technique (Muskhelishvili, 1953). These solutions read, respectively, as (Appendix B):

$$\theta'(z) = -\frac{k_{\mathrm{I}} + k_{\mathrm{II}}}{2\pi k_{\mathrm{I}}k_{\mathrm{II}}}x(z)\left[\int_a^b \frac{x_+^{-1}(x_1)h_0(x_1)}{x_1 - z}\mathrm{d}x_1 + P(z)\right], \tag{33}$$

$$\Phi'(z) = \frac{1}{2\pi}x(z)\left[\int_a^b \frac{x_+^{-1}(x_1)}{x_1 - z}N^{-1}[p(x_1) + \varrho_1\theta_+(x_1) + \varrho_2\theta_-(x_1)]\mathrm{d}x_1 + Q(z)\right], \tag{34}$$

where $P(z)$ and $Q(z)$ are polynomial of z with degree less than one,

$$x(z) = \frac{1}{\sqrt{(z-a)(z-b)}}, \quad \mathrm{X}(z) = \mathbf{v}x(z)\Delta(z;\epsilon), \quad \Delta(z;\epsilon) = \mathrm{diag}\left[\left(\frac{z-b}{z-a}\right)^{\mathrm{i}\epsilon}, \left(\frac{z-b}{z-a}\right)^{-\mathrm{i}\epsilon}, 1\right] \tag{35}$$

and

$$\mathbf{v} = [v_1, v_2, v_3], \tag{36}$$

in which, v_j ($j = 1,2,3$) is the eigenvectors of equation:

$$(N + \mathrm{e}^{2\pi\mathrm{i}\delta}\overline{N})\mathbf{v} = \mathbf{0}. \tag{37}$$

The matrix N can be expressed in terms of a symmetric matrix D and anti-symmetric matrix W as Ting (1986):

$$N^{-1} = D - \mathrm{i}W, \quad D = L_1^{-1} + L_2^{-1}, \quad W = S_1L_1^{-1} - S_2L_2^{-1}. \tag{38}$$

An explicit solution to eigenvalues of Eq. (37) is

$$\delta_1 = \frac{1}{2} + \mathrm{i}\epsilon, \quad \delta_2 = \frac{1}{2} - \mathrm{i}\epsilon, \quad \delta_2 = \frac{1}{2}; \quad \text{with } \epsilon = \frac{1}{2\pi}\log\left[\frac{1+\gamma}{1-\gamma}\right], \quad \gamma = \left[-\frac{1}{2}\mathrm{tr}(D^{-1}W)^2\right]^{\frac{1}{2}}. \tag{39}$$

It can be seen that once the applied loading $h_0(x_1)$ and $p(x_1)$ is given, then the solution to the functions $\theta(z)$ and $\Phi(z)$, hence fields functions $\chi_j(z)$ and $\phi_j(z)$ ($j =$ 'I' and 'II') can be found. Therefore, a general solution to the thermo-elastic interface crack problem of dissimilar bi-media is then obtained. The stresses $\sigma_{i2} = \varphi'$ ahead of the interface crack read

$$[\sigma_{12}, \sigma_{22}, \sigma_{32}]^{\mathrm{T}} = \varphi' = N^*\Phi'(x_1) - e^*\theta(x_1), \quad x_1 < a \text{ or } b < x_1, \tag{40}$$

where

$$N^* = \mathrm{i}(N + \overline{N}), \quad e_3 = \frac{k_{\mathrm{II}}D_{\mathrm{I}} + k_{\mathrm{I}}D_{\mathrm{II}}}{k_{\mathrm{I}} + k_{\mathrm{I}}}, \\ e^* = \mathrm{i}(Ne_1 + \overline{N}\overline{e}_1) - (NM_{\mathrm{I}}^{-1}e_2 - \overline{NM_{\mathrm{I}}^{-1}}\overline{e}_2) + e_2 + e_3 \tag{41}$$

and the crack open displacements (COD) can be delivered after some tedious manipulation:

$$\Delta\mathbf{u} = \mathbf{u}_+^{\mathrm{I}}(x_1) - \mathbf{u}_-^{\mathrm{II}}(x_1) = \Phi_+(x_1) - \Phi_-(x_1), \quad a \leqslant x_1 \leqslant b. \tag{42}$$

2.2. Solution for the constant applied loading

If the applied loading on the crack interface is constant, i.e. $h_0(x_1) = h_0$ and $p(x_1) = p_0$, then by contour integration Eq. (1) leads to

$$\theta'(z) = -\mathrm{i}\frac{(k_{\mathrm{I}} + k_{\mathrm{II}})h_0}{2k_{\mathrm{I}}k_{\mathrm{II}}}\left[1 - \frac{z - (a+b)/2}{\sqrt{(z-a)(z-b)}}\right]. \tag{43}$$

Integration of Eq. (43) gives

$$\theta(z) = -\mathrm{i}\frac{(k_{\mathrm{I}} + k_{\mathrm{II}})h_0}{2k_{\mathrm{I}}k_{\mathrm{II}}}\left[z - \sqrt{(z-a)(z-b)}\right], \tag{44}$$

where the integral constant is dropped. The stress function can be found from (34) and it reads:

$$\Phi'(z) = v[\phi_1(z)v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_0) + \phi_2(z)v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_1^*) + \phi_3(z)v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*)], \tag{45}$$

where

$$\phi_1(z) = I - x(z)\Delta(z;\epsilon)[\Xi(z) + \Pi_1], \\ \phi_2(z) = \Xi(z) - x(z)\Delta(z;\epsilon)[\Xi(z^2) + \Pi_1\Xi(z) - \Pi_2] + x(z)\Pi_5, \\ \phi_3(z) = x^{-1}(z) - x(z)\Delta(z;\epsilon)[\Xi(z^2) - \Pi_3\Xi(z) + \Pi_4] + x(z)\Pi_6 \tag{46}$$

and Π_k ($k = 1$ to 6) are defined in (113).

If the constant which only contributes rigid body motion is omitted, integration of the above function gives (Appendix B):

$$\Phi(z) = v[\Xi(z) - x^{-1}(z)\Delta(z;\epsilon)]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_0) + v[\Xi(z^2) - x^{-1}\Delta(z;\epsilon)\Xi(z) - Y_1(z;\epsilon) \\ - Y_2(z;\epsilon)\Pi_2]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_1^*) + v[Y_3(z) - x^{-1}\Delta(z;\epsilon)(\Xi(z) - \widetilde{\Pi}_1) - Y_1(z;\epsilon) \\ - Y_2(z;\epsilon)\widetilde{\Pi}_2]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*), \tag{47}$$

where

$$\widetilde{\Pi}_1 = \mathrm{diag}\left[\frac{a+b}{2} + (b-a)\mathrm{i}\epsilon,\ \frac{a+b}{2} - (b-a)\mathrm{i}\epsilon,\ \frac{a+b}{2}\right], \\ \widetilde{\Pi}_2 = \mathrm{diag}\left[\frac{b^2-a^2}{2}\mathrm{i}\epsilon - (1+4\epsilon^2)\frac{(b-a)^2}{2},\ -\frac{b^2-a^2}{2}\mathrm{i}\epsilon - (1+4\epsilon^2)\frac{(b-a)^2}{2},\ -\frac{(b-a)^2}{2}\right], \tag{48}$$

$\Xi(z)$, $Y_1(z;\epsilon)$, $Y_2(z;\epsilon)$ and $Y_3(z)$ are matrix functions defined in Appendix B. Once the temperature potential and stress functions are found, the heat flux and stress field for this bi-media can be readily obtained. Herein is given the heat flux for the upper medium of this bi-material:

$$h_1^c(x_1,x_2) = -\mathrm{Re}\left[\left(1-\frac{z-\frac{a+b}{2}}{\sqrt{(z-a)(z-b)}}\right)\tau\right]h_0,$$
$$h_2^c(x_1,x_2) = \mathrm{Re}\left[1-\frac{z-\frac{a+b}{2}}{\sqrt{(z-a)(z-b)}}\right]h_0 \tag{49}$$

and the stress fields for the upper medium read as

$$[\sigma_{11},\sigma_{21},\sigma_{31}]^{\mathrm{T}}_{1c} = -2\mathrm{Re}[\mathrm{i}Nv \ll p_\alpha \gg v^{-1}\Phi'(z) - \mathrm{i}\overline{D_c}\tau\theta(z)],$$
$$[\sigma_{11},\sigma_{21},\sigma_{31}]^{\mathrm{T}}_{2c} = 2\mathrm{Re}[\mathrm{i}N\Phi'(z) - \mathrm{i}\overline{D_c}\theta(z)], \tag{50}$$

where

$$\overline{D_c} = \mathrm{i}Ne_1 + N\overline{M}_{\mathrm{II}}^{-1}e_2 - D_I\frac{k_{\mathrm{II}}}{k_I+k_{\mathrm{II}}}. \tag{51}$$

The COD for this case can then be expressed as

$$\Delta\mathbf{u}(x_1) = 4\sqrt{(x_1-a)(b-x_1)}\mathrm{cosch}(\epsilon\pi)\{\mathbf{u}_1(x_1,\epsilon)[p_0 + x_1(p_1^*+p_2^*) - \widetilde{\Pi}_1p_2^*] + \frac{a+b-2x_1}{8}(N+\overline{N})^{-1}p_2^*\}, \tag{52}$$

where

$$\mathbf{u}_1(x_1,\epsilon) = \mathbf{v}\mathrm{diag}\left[\left(\frac{b-x_1}{x_1-a}\right)^{\mathrm{i}\epsilon},\left(\frac{b-x_1}{x_1-a}\right)^{-\mathrm{i}\epsilon},\mathrm{cosch}^{-1}(\epsilon\pi)\right]\mathbf{v}^{-1}(N+\overline{N})^{-1}. \tag{53}$$

The traction ahead of the crack tip may then given by

$$\begin{aligned}\mathbf{t}(x_1) &= [\sigma_{12},\sigma_{22},\sigma_{32}]^{\mathrm{T}} = N^*\Phi'(x_1) - e^*\theta(x_1)\\ &= \frac{N^*}{\sqrt{(x_1-a)(x_1-b)}}v\Big\{[\sqrt{(x_1-a)(x_1-b)}\mathrm{I} - \Delta(x_1;\epsilon)(\Xi(x_1)+\Pi_1)]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_0)\\ &\quad+[x_1\sqrt{(x_1-a)(x_1-b)}\mathrm{i} - \Delta(x_1;\epsilon)(\Xi(x_1^2)+x_1\Pi_1-\Pi_2)+\Pi_5]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_1^*)\\ &\quad+[(x_1-a)(x_1-b)\mathrm{i} - \Delta(x_1;\epsilon)(\Xi(x_1^2)-x_1\Pi_3+\Pi_4)+\Pi_6]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*)\Big\}\\ &\quad- e^*[x_1-\sqrt{(x_1-a)(x_1-b)}]h_0^*,\end{aligned} \tag{54}$$

the notations Π_1, Π_2, Π_3, Π_4, Π_5 and Π_6 are defined in Appendix B and $\mathrm{I} = \mathrm{diag}[1, 1, 1]$. The conventional Stress Intensity Factors (**SIF**s) ahead the crack tip such as for $x_1 = b$ may be expressed as

$$\begin{aligned}[\mathbf{K}_{\mathrm{II}},\mathbf{K}_{\mathrm{I}},\mathbf{K}_{\mathrm{III}}]^{\mathrm{T}} &= \lim_{x_1\to b}\sqrt{2\pi(x_1-b)}[\sigma_{12},\sigma_{22},\sigma_{32}]^{\mathrm{T}}\\ &= \sqrt{2\pi(b-a)}N^*v\lim_{x_1\to b}\Delta(x_1;\epsilon)[\mathrm{k}_1v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_0) + \mathrm{k}_2v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_1^*)\\ &\quad+\mathrm{k}_3v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*)],\end{aligned} \tag{55}$$

where

$$\begin{aligned}\mathrm{k}_1 &= -\mathrm{diag}\left[\frac{1}{2}+\mathrm{i}\epsilon,\frac{1}{2}-\mathrm{i}\epsilon,\frac{1}{2}\right],\\ \mathrm{k}_2 &= (b-a)\mathrm{diag}\left[\epsilon^2-\frac{b+a}{4(b-a)}-b\mathrm{i}\epsilon,\epsilon^2-\frac{b+a}{4(b-a)}+b\mathrm{i}\epsilon,-\frac{b+a}{4(b-a)}+\frac{1}{8}\right],\\ \mathrm{k}_3 &= (b-a)\mathrm{diag}[0.375+\epsilon^2+2\mathrm{i}\epsilon,0.375+\epsilon^2-2\mathrm{i}\epsilon,-0.25].\end{aligned} \tag{56}$$

Now the energy release rate G_0 can also be calculated for this interface crack propagation. Assuming the crack grow at crack tip 'b' to '$b+\delta b$', G_0 can be found from Eqs. (42), (47) and (54) as

$$G_0 = \lim_{\delta b\to 0}\frac{1}{2\delta b}\int_0^{\Delta b}\delta\mathbf{u}^{\mathrm{T}}(x_1-\delta b)\mathbf{t}(x_1)\mathrm{d}x_1. \tag{57}$$

For the simple case of the two media are identical, the explicit expressions for SIFs and the energy release rate can be obtained, respectively, as

$$\begin{aligned}&[\mathbf{K}_{\mathrm{II}},\mathbf{K}_{\mathrm{I}},\mathbf{K}_{\mathrm{III}}]^{\mathrm{T}} = -\mathrm{Re}\{\sqrt{2\pi(b-a)}[\mathrm{k}_1 p_0 + \mathrm{k}_2 p_1^* + \mathrm{k}_3 p_2^*]\},\\ &G_0 = \mathrm{Re}\left\{\frac{\pi(b-a)}{2}[p_0^{\mathrm{T}}L^{-1}p_0 + (b-a)p_0^{\mathrm{T}}L^{-1}\widetilde{p}_1^* + p_0^{\mathrm{T}}L^{-1}e_1^* h_0^* + bp_1^{*\mathrm{T}}L^{-1}p_0 + bp_1^{*\mathrm{T}}L^{-1}\hat{p}_1^*/4]\right\},\end{aligned} \tag{58}$$

where

$$p_1^{*\mathrm{T}} = p_1^*\mathrm{diag}[1,1,1];\quad \widetilde{p}_1 = p_1^*\mathrm{diag}\left[1,1,\frac{3}{2}\right],\quad \hat{p}_1^* = p_1^*\mathrm{diag}\left[b+a,b+a,\frac{b+3a}{2}\right]. \tag{59}$$

If there is no applied mechanical loading, i.e $p_0 = [0, 0, 0]^{\mathrm{T}}$, then Eq. (58) can be expressed as:

$$G_0 = \frac{\pi b(b-a)}{8}p_1^{*\mathrm{T}}L^{-1}\hat{p}_1^*. \tag{60}$$

So far in this section, a solution as well as the method leading to the solution for a crack in a thermo-mechanically loaded anisotropic medium was presented in details. And it can be seen that the general solution given here lays the foundation for the study of the branched thermo-elastic crack phenomena.

3. Green's functions for thermo-elastic dislocations in anisotropic bi-media

When a dislocation (Stroh, 1958) is introduced into one of the elastic bi-media under thermal loading, a temperature discontinuity (also called heat vortex, Dundurs and Comninou, 1979) is induced across the cut plane associated with the conventional (or mechanical) dislocation. This concept of heat vortex first appeared in literature several decades ago and has been studied by many authors, such as Sturla and Barber (1988). But most of the functions of displacement and stress fields due to the heat vortex cannot be directly extended to the dissimilar anisotropic media. To overcome this difficulty, mixed terms are adopted in the expressions for displacement and stress functions. The functions of the heat vortex may be assumed for dissimilar anisotropic bi-media (Fig. 2) as

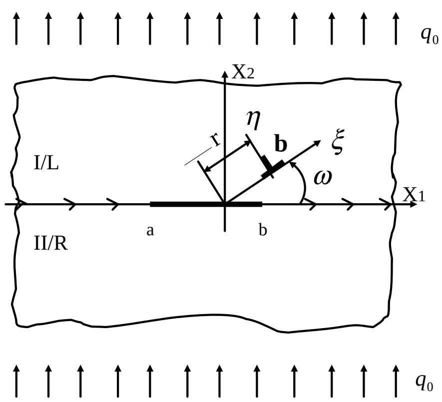

Fig. 2. A thermo-elastic dislocation in dissimilar anisotropic bi-medium.

$$T_{\mathrm{I}}^{\mathrm{d}} = 2\,\mathrm{Re}[q_{0\tau}\log(z_\tau - z_{\tau 0}) + q_{1\tau}\log(z_\tau - \bar{z}_{\tau 0})], \quad z \in L;$$
$$T_{\mathrm{II}}^{\mathrm{d}} = 2\,\mathrm{Re}[q_{2\tau}\log(z_\tau - z_{\tau 0})], \quad z \in R. \tag{61}$$

The corresponding heat flux h_2 can then be expressed as (Sturla and Barber, 1988):

$$h_{2\mathrm{I}}^{\mathrm{d}} = 2k_{\mathrm{I}}\mathrm{Im}\left[\frac{q_{0\tau}}{z_\tau - z_{\tau 0}} + \frac{q_{1\tau}}{z_\tau - \bar{z}_{\tau 0}}\right], \quad z \in L; \quad h_{2\mathrm{II}}^{\mathrm{d}} = 2k_{\mathrm{II}}\mathrm{Im}\left[\frac{q_{2\tau}}{z_\tau - z_{\tau 0}}\right], \quad z \in R, \tag{62}$$

where $q_{0\tau} = \frac{T}{4\pi i}$, $q_{1\tau}$ and $q_{2\tau}$ are constants to be determined. The displacement and stress functions may then take the form

$$\begin{aligned}
\mathbf{u}_{\mathrm{I}}^{\mathrm{d}} = {}& 2\mathrm{Re}[A_{\mathrm{I}} \ll \log(z_\alpha - z_{d0}) \gg q_{d0}] + \left[\sum_{k=1}^{3} A_{\mathrm{I}} \ll \log(z_\alpha - \bar{z}_{d0k}) \gg q_{1k}\right] \\
& + 2\mathrm{Re}[A_{\mathrm{I}} \ll (\log(z_\alpha - z_{\tau 0}) - 1)(z_\alpha - z_{\tau 0}) \gg q_{1d\tau}] \\
& + 2\mathrm{Re}[C_{\mathrm{I}}(q_{0\tau}(\log(z_\tau - z_{\tau 0}) - 1)(z_\tau - z_{\tau 0}) + q_{1\tau}(\log(z_\tau - \bar{z}_{\tau 0}) - 1)(z_\tau - \bar{z}_{\tau 0}))], \\
\phi_{\mathrm{I}}^{\mathrm{d}} = {}& 2\mathrm{Re}[B_{\mathrm{I}} \ll \log(z_\alpha - z_{d0}) \gg q_{d0}] + 2\mathrm{Re}\left[\sum_{k=1}^{3} B_{\mathrm{I}} \ll \log(z_\alpha - \bar{z}_{d0k}) \gg q_{1k}\right] \\
& + 2\mathrm{Re}[B_{\mathrm{I}} \ll (\log(z_\alpha - z_{\tau 0}) - 1)(z_\alpha - z_{\tau 0}) \gg q_{1d\tau}] \\
& + 2\mathrm{Re}[D_{\mathrm{I}}(q_{0\tau}(\log(z_\tau - z_{\tau 0}) - 1)(z_\tau - z_{\tau 0}) + q_{1\tau}(\log(z_\tau - \bar{z}_{\tau 0}) - 1)(z_\tau - \bar{z}_{\tau 0})]
\end{aligned} \tag{63}$$

for upper half-space ($x_2 > 0$) and

$$\begin{aligned}
\mathbf{u}_{\mathrm{II}}^{\mathrm{d}} = {}& 2\mathrm{Re}\left[\sum_{k=1}^{3} A_{\mathrm{II}} \ll \log(z_\alpha - z_{d0k}) \gg q_{2k}\right] + 2\mathrm{Re}[A_{\mathrm{II}} \ll (\log(z_\alpha - \bar{z}_{\tau 0}) - 1)(z_\alpha - \bar{z}_{\tau 0}) \gg q_{2d\tau}] \\
& + 2\mathrm{Re}[C_{\mathrm{II}}(\log(z_\tau - z_{\tau 0}) - 1)(z_\tau - z_{\tau 0})q_2\tau], \\
\phi_{\mathrm{II}}^{\mathrm{d}} = {}& 2\mathrm{Re}\left[\sum_{k=1}^{3} B_{\mathrm{II}} \ll \log(z_\alpha - z_{d0k}) \gg q_{2k}\right] + 2\mathrm{Re}[B_{\mathrm{II}} \ll (\log(z_\alpha - \bar{z}_{\tau 0}) - 1)(z_\alpha - \bar{z}_{\tau 0}) \gg q_{2d\tau}] \\
& + 2\mathrm{Re}[D_{\mathrm{II}}(\log(z_\tau - z_{\tau 0}) - 1)(z_\tau - z_{\tau 0})q_2\tau]
\end{aligned} \tag{64}$$

for lower half-space ($x_2 < 0$), where $q_{d0} = \frac{1}{2\pi i}B_{\mathrm{I}}^{\mathrm{T}}\mathbf{b}$ (Barber and Comninou, 1982). It should be mentioned the mixed terms $\ll (\log(z_\alpha - z_{\tau 0}) - 1)(z_\alpha - z_{\tau 0}) \gg$ and $\ll (\log(z_\alpha - \bar{z}_{\tau 0}) - 1)(z_\alpha - \bar{z}_{\tau 0}) \gg$ were introduced to reflect the interaction between the heat vortex and the conventional dislocation due to the mismatch of the properties of the upper and lower media. This is very important in order to ensure the continuity of the displacements and tractions along the interface of the dissimilar bi-materials. Substituting Eqs. (61)–(64) into the boundary conditions along the interface,

$$\begin{aligned}
& T_{\mathrm{I}}^{\mathrm{d}}(x_1, x_2 = 0^+) = T_{\mathrm{II}}^{\mathrm{d}}(x_1, x_2 = 0^-), \quad h_{2\mathrm{I}}^{\mathrm{d}}(x_1, x_2 = 0^+) = h_{2\mathrm{II}}^{\mathrm{d}}(x_1, x_2 = 0^-), \\
& \mathbf{u}_{\mathrm{I}}^{\mathrm{d}}(x_1, x_2 = 0^+) = \mathbf{u}_{\mathrm{II}}^{\mathrm{d}}(x_1, x_2 = 0^-), \quad \phi_{\mathrm{I}}'^{\mathrm{d}}(x_1, x_2 = 0^+) = \phi_{\mathrm{II}}'^{\mathrm{d}}(x_1, x_2 = 0^-),
\end{aligned} \tag{65}$$

one can obtain (Appendix C):

$$\begin{aligned}
& q_{1\tau} = \frac{k_{\mathrm{I}} - k_{\mathrm{II}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}\bar{q}_{0\tau}, \quad q_{2\tau} = \frac{2k_{\mathrm{I}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}q_{0\tau}, \\
& B_{\mathrm{I}}q_{1k} = N(-N^{-1} + 2L_{\mathrm{I}}^{-1})\overline{B}_{\mathrm{I}}I_k\bar{q}_{d0}, \quad B_{\mathrm{II}}q_{2k} = 2\overline{N}L_{\mathrm{I}}^{-1}B_{\mathrm{I}}I_k q_{d0}, \\
& B_{\mathrm{I}}q_{1d\tau} = N[\overline{M}_{\mathrm{II}}^{-1}D + \mathrm{i}C]q_{0\tau}, \quad B_{\mathrm{II}}q_{2d\tau} = -\overline{N}[\overline{M}_{\mathrm{I}}^{-1}\overline{D} + \mathrm{i}\overline{C}]q_{0\tau}.
\end{aligned} \tag{66}$$

The heat flux and stress fields can then be readily calculated. Following are these quantities for the upper medium,

$$h_{1\mathrm{I}}^{\mathrm{td}} = -2k_{\mathrm{I}}\mathrm{Im}\left[\frac{q_{0\tau}}{z_\tau - z_{\tau 0}}\tau + \frac{q_{1\tau}}{z_\tau - \bar{z}_{\tau 0}}\tau\right],$$
$$h_{2\mathrm{I}}^{\mathrm{td}} = 2k_{\mathrm{I}}\mathrm{Im}\left[\frac{q_{0\tau}}{z_\tau - z_{\tau 0}} + \frac{q_{1\tau}}{z_\tau - \bar{z}_{\tau 0}}\right] \tag{67}$$

and

$$[\sigma_{11},\sigma_{21},\sigma_{31}]_{\mathrm{I}1}^{\mathrm{tdT}} = -2\mathrm{Re}\sum_{k=1}^{3}\left[B_{\mathrm{I}} \ll \frac{p_\alpha}{z_\alpha - z_{d0k}} \gg I_k q_0 + B_{\mathrm{I}} \ll \frac{p_\alpha}{z_\alpha - \bar{z}_{d0k}} \gg q_{1k}\right]$$
$$+ 2\mathrm{Re}[B_{\mathrm{I}} \ll p_\alpha \log(z_\alpha - z_{\tau 0}) \gg q_{1d\tau} + D_{\mathrm{I}}(\tau\log(z_\tau - z_{\tau 0})q_{0\tau} + \tau\log(z_\tau - \bar{z}_{\tau 0})q_{1\tau})],$$
$$[\sigma_{12},\sigma_{22},\sigma_{32}]_{\mathrm{I}2}^{\mathrm{td}T} = 2\mathrm{Re}\sum_{k=1}^{3}\left[B_{\mathrm{I}} \ll \frac{1}{z_\alpha - z_{d0k}} \gg I_k q_0 + B_{\mathrm{I}} \ll \frac{1}{z_\alpha - \bar{z}_{d0k}} \gg q_{1k}\right]$$
$$+ 2\mathrm{Re}[B_{\mathrm{I}} \ll \log(z_\alpha - z_{\tau 0}) \gg q_{1d\tau} + D_{\mathrm{I}}(\log(z_\tau - z_{\tau 0})q_{0\tau} + \log(z_\tau - \bar{z}_{\tau 0})q_{1\tau})]. \tag{68}$$

The heat flux and tractions along the interface are, respectively:

$$h_2^{\mathrm{d}}(x_1) = \frac{4k_{\mathrm{I}}k_{\mathrm{II}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}\mathrm{Im}\left[\frac{q_{0\tau}}{x_1 - z_{\tau 0}}\right] \tag{69}$$

and

$$\mathbf{t}_{d\tau} = [\sigma_{12},\sigma_{22},\sigma_{32}]_{d\tau}^{\mathrm{T}}$$
$$= 2\mathrm{Re}\left\{\sum_{k=1}^{3}\left[\frac{2}{x_1 - z_{d0k}}\overline{N}L_{\mathrm{I}}^{-1}B_{\mathrm{I}}I_k q_{d0}\right] - \left[\log(x_1 - \bar{z}_{\tau 0})\overline{N}(\overline{M}_{\mathrm{I}}^{-1}\overline{D} + \mathrm{i}\overline{C}) - \log(x_1 - z_{\tau 0})\frac{2k_{\mathrm{I}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}D_{\mathrm{II}}\right]q_{0\tau}\right\}$$
$$= 2\mathrm{Re}\left\{\sum_{k=1}^{3}\left[\frac{2}{x_1 - z_{d0k}}\overline{N}L_{\mathrm{I}}^{-1}B_{\mathrm{I}}I_k q_{d0}\right] + \log(x_1 - z_{\tau 0})[N(M_{\mathrm{I}}^{-1}D - \mathrm{i}C) + \frac{2k_{\mathrm{I}}}{k_{\mathrm{I}} + k_{\mathrm{II}}}D_{\mathrm{II}}]q_{0\tau}\right\}, \tag{70}$$

where the relationship $\mathrm{Re}[1/(x_1 - z_{d0k})] = \mathrm{Re}[1/(x_1 - \bar{z}_{d0k})]$ and $\mathrm{Re}[\log(x_1 - z_{\tau 0}] = \mathrm{Re}[\log(x_1 - \bar{z}_{\tau 0}]$ are used.

4. Thermo-elastic interaction between the interface crack and the dislocations

Replacing the $h_0(x_1)$ of Eq. $(33)_1$ with $-h_2^{\mathrm{d}}(x_1)$ of Eq. 69, one can obtain a closed form solution for the interaction temperature potential function, and this reads:

$$\theta'_{\mathrm{int}}(z) = \frac{T}{4\pi}[y(z, z_{\tau 0}) + y(z, \bar{z}_{\tau 0})], \tag{71}$$

where

$$y(z, z_{\tau 0}) = \frac{1}{z - z_{\tau 0}}[1 - x(z)x^{-1}(z_{\tau 0})] - x(z). \tag{72}$$

Integrating Eq. (71) and dropping some constants yields

$$\theta_{\rm int}(z) = \frac{T}{4\pi}[\widetilde{y}(z, z_{\tau 0}) + \widetilde{y}(z, \bar{z}_{\tau 0})] \tag{73}$$

with

$$\widetilde{y}(z, z_{\tau 0}) = \log\left[\frac{x^{-1}(z) + x^{-1}(z_{\tau 0}) + \left(z_{\tau 0} - \frac{a+b}{2}\right)(z - z_{\tau 0})x(z_{\tau 0})}{z - \frac{a+b}{2} + \sqrt{(z-a)(z-b)}}\right]. \tag{74}$$

It can be seen that the interaction thermal potential function is not singular at the point $z = z_{\tau 0}$. Comparing with the contribution from the term $\frac{1}{z - z_{\tau 0}}$ for the onset of interface crake branching, the influence of function $\theta_{\rm int}(z)$ on the interaction stress functions, which can be obtained by replacing $p(x_1)$ of Eq. $(33)_2$ with $-\mathbf{t}_{d\tau}$ of Eq. (70), can be ignored. Therefore, the interaction stress functions can be obtained as

$$\Phi'_{\rm int}(z) = \sum_{k=1}^{3}[\mathbf{v}\mathbf{Y}_k(z, z_{d0k}; \epsilon)\mathbf{v}^{-1}(N + \overline{N})^{-1}\mathscr{A}_k - \mathbf{v}\mathbf{Y}_k(z, \bar{z}_{d0k}; \epsilon)\mathbf{v}^{-1}(N + \overline{N})^{-1}\overline{\mathscr{A}}_k]\mathbf{b}, \tag{75}$$

where

$$\mathbf{Y}_k(z, z_{d0k}; \epsilon) = \ll \frac{1}{z - z_{d0k}} \gg \left[\mathrm{i} - \sqrt{\frac{(z_{d0k} - a)(z_{d0k} - b)}{(z-a)(z-b)}}\varDelta(z;\epsilon)\varDelta^{-1}(z_{d0k};\epsilon)\right] - \frac{\varDelta(z;\epsilon)}{\sqrt{(z-a)(z-b)}}, \tag{76}$$

$$\mathscr{A}_k = \overline{N}L_{\rm I}^{-1}B_{\rm I}I_kB_{\rm I}^{\rm T}/\pi,$$

and the following notation is employed:

$$\frac{\varDelta(z;\epsilon)}{\sqrt{(z-a)(z-b)}} = \mathrm{diag}\left[(z_1 - b)^{-\frac{1}{2}+\mathrm{i}\epsilon}(z_1 - a)^{-\frac{1}{2}-\mathrm{i}\epsilon}, (z_2 - b)^{-\frac{1}{2}-\mathrm{i}\epsilon}(z_2 - a)^{-\frac{1}{2}+\mathrm{i}\epsilon}, (z_3 - b)^{-\frac{1}{2}}(z_3 - a)^{-\frac{1}{2}}\right]. \tag{77}$$

By employing L'Hospital principle, one can easily show that the $y(z, z_{\tau 0})$ and $\mathbf{Y}_k(z, z_{d0k}; \epsilon)$ is not singular when $z \to \mathrm{z}_\tau$ and $z \to z_{d0k}$, respectively.

The heat flux and stress fields induced by the interaction for the upper medium can then be written, respectively, as

$$h_1^{\rm int} = -2\frac{k_{\rm I}k_{\rm II}}{k_{\rm I} + k_{\rm II}}\mathrm{Re}[\tau\theta'_{\rm int}(z)], \quad h_2^{\rm int} = 2\frac{k_{\rm I}k_{\rm II}}{k_{\rm I} + k_{\rm II}}\mathrm{Re}[\theta'_{\rm int}(z)] \tag{78}$$

and

$$\begin{aligned}[\sigma_{11}, \sigma_{21}, \sigma_{31}]_{\rm I}^{\rm int\,T} &= -2\mathrm{Re}[\mathrm{i}Nv \ll p_\alpha \gg v^{-1}\Phi'_{\rm int}(z) - \mathrm{i}\overline{D_{\rm int}}\tau\theta_{\rm int}(z)], \\ [\sigma_{12}, \sigma_{22}, \sigma_{32}]_{\rm I}^{\rm int\,T} &= 2\mathrm{Re}[\mathrm{i}N\Phi'_{\rm int}(z) - \mathrm{i}\overline{D_{\rm int}}\theta_{\rm int}(z)],\end{aligned} \tag{79}$$

where $\overline{D_{\rm int}} = \overline{D_c}$.

5. Thermo-elastic interface crack branching in anisotropic bi-media

A main crack located at the $a < x_1 < b$, $x_2 = 0$ of coordinate system (x_1, x_2, x_3) is assumed to branch into $x_2 > 0$ (or $x_2 < 0$) at an angle $\theta = \omega$ shown in Fig. 3, in which a new coordinate system (ξ, η, x_3) is

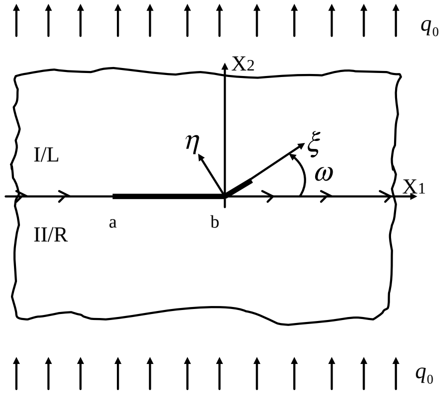

Fig. 3. A branched thermo-elastic interface crack in dissimilar anisotropic.

introduced for the sake of convenience. Similarly to the conditions for the main crack, the boundary conditions for this branched portion read in this new coordinate system as:

$$\begin{aligned}&h_2(\xi,0^+)=-h(\xi),\quad h_2(\xi,0^-)=-h(\xi);\\&[\sigma_{\xi\eta}(\xi,0^+),\sigma_{\eta\eta}(\xi,0^+),\sigma_{3\eta}(\xi,0^+)]^{\mathrm{T}}=-p(\xi);\quad[\sigma_{\xi\eta}(\xi,0^-),\sigma_{\eta\eta}(\xi,0^-),\sigma_{3\eta}(\xi,0^-)]^{\mathrm{T}}=-p(\xi).\end{aligned}\tag{80}$$

If the applied thermo-mechanical loading at infinity is constant, then:

$$h(\xi)=h_0\cos(\omega);\quad p(\xi)=\begin{vmatrix}\cos(2\omega), & \frac{1}{2}\sin(2\omega) & 0\\ -\sin(2\omega), & \cos^2(\omega) & 0\\ 0, & 0 & \cos(\omega)\end{vmatrix}p_0,\tag{81}$$

where vector $p_0=[\sigma_{12},\sigma_{22},\sigma_{32}]^{\mathrm{T}}$ is the constant applied traction at infinity.Now let us consider the total heat flux and traction at any point on the plane $\eta=0$, i.e. $\theta=\omega$ in the polar coordinates system (r,θ,x_3), then by superposition:

$$\begin{aligned}&h_2^{\mathrm{tot}}(\xi,0)=h_\theta^{\mathrm{c}}(r,\omega)+h_\theta^{\mathrm{int}}(r,\omega)+h_\theta^{\mathrm{td}}(r,\omega),\\&\mathbf{t}^{\mathrm{tot}}(\xi,0)=\mathbf{t}_\theta^{\mathrm{c}}(r,\omega)+\mathbf{t}_\theta^{\mathrm{int}}(r,\omega)+\mathbf{t}_\theta^{\mathrm{td}}(r,\omega),\end{aligned}\tag{82}$$

where the superscript ‘c’ and ‘td’ denote the corresponding fields induced by the main crack and the thermal-mechanical dislocations, respectively; ‘int’ denotes the fields induced by the interaction between the crack and the dislocation and ‘tot’ is the summation from all contributions. It would be more convenient for the calculation if the terms on the right sides of the Eqs. (82), expressed in the coordinate system (x_1,x_2,x_3), are transformed into the corresponding quantities in the coordinate system (r,θ,x_3) or the system (ξ,η,x_3). Following is the transformation relationship

$$\begin{aligned}&h=h_2\cos(\omega)-h_1\sin(\omega),\\&\mathbf{t}=\Omega_2(\omega)[\sigma_{12},\sigma_{22},\sigma_{32}]^{\mathrm{T}}-\Omega_1(\omega)[\sigma_{11},\sigma_{21},\sigma_{31}]^{\mathrm{T}},\end{aligned}\tag{83}$$

where

$$\Omega_2(\theta)=\begin{vmatrix}\cos^2(\theta) & \frac{1}{2}\sin(2\theta) & 0\\ -\frac{1}{2}\sin(2\theta) & \cos^2(\theta) & 0\\ 0 & 0 & \cos(\theta)\end{vmatrix},\quad \Omega_1(\theta)=\begin{vmatrix}\frac{1}{2}\sin(2\theta) & \sin^2(\theta) & 0\\ -\sin^2(\theta) & \frac{1}{2}\sin(2\theta) & 0\\ 0 & 0 & \sin(\theta)\end{vmatrix}\tag{84}$$

and σ_{i1} and σ_{i2} are stresses measured in system (x_1, x_2, x_3) and defined by Eq. (106); h_1 and h_2 are heat flux measured in system (x_1, x_2, x_3) and defined by Eq. (107). Then, each term of the right hand side of Eq. (82) can be easily expressed in terms of the temperature potential functions and stress functions obtained in previous sections. If let $\mu = \cos(\omega) + \tau\sin(\omega)$ and $\zeta = \cos(\omega) + p_\alpha\sin(\omega)$, then $z_\tau = r\mu$, $z_{\tau 0} = r_0\mu$, $z_\alpha = r\zeta$ and $z_{\alpha 0} = r_0\zeta$. Therefore, one has:

$$\begin{aligned} h_\theta^{\mathrm{c}}(r,\omega) &= h_2^{\mathrm{c}}(r\mu)\cos(\omega) - h_1^{\mathrm{c}}(r\mu)\sin(\omega), \\ h_\theta^{\mathrm{int}}(r,\omega) &= h_2^{\mathrm{int}}(r\mu)\cos(\omega) - h_1^{\mathrm{int}}(r\mu)\sin(\omega), \\ h_\theta^{\mathrm{td}}(r,\omega) &= h_2^{\mathrm{td}}(r\mu)\cos(\omega) - h_1^{\mathrm{td}}(r\mu)\sin(\omega) \end{aligned} \tag{85}$$

and

$$\begin{aligned} \mathbf{t}_\theta^{\mathrm{c}}(r,\omega) &= \Omega_2(\omega)[\sigma_{12},\sigma_{22},\sigma_{32}]_c^{\mathrm{T}} - \Omega_1(\omega)[\sigma_{11},\sigma_{21},\sigma_{31}]_c^{\mathrm{T}}, \\ \mathbf{t}_\theta^{\mathrm{int}}(r,\omega) &= \Omega_2(\omega)[\sigma_{12},\sigma_{22},\sigma_{32}]_{\mathrm{int}}^{\mathrm{T}} - \Omega_1(\omega)[\sigma_{11},\sigma_{21},\sigma_{31}]_{\mathrm{int}}^{\mathrm{T}}, \\ \mathbf{t}_\theta^{\mathrm{td}}(r,\omega) &= \Omega_2(\omega)[\sigma_{12},\sigma_{22},\sigma_{32}]_{\mathrm{td}}^{\mathrm{T}} - \Omega_1(\omega)[\sigma_{11},\sigma_{21},\sigma_{31}]_{\mathrm{td}}^{\mathrm{T}}. \end{aligned} \tag{86}$$

Without loss of generality, it can be assumed that the interface crack branches into the upper media. The branched portion of the crack can be modelled by the continuous distribution of the dislocations with density $T_0(r_0) = -\mathrm{d}T_0(r_0)/\mathrm{d}r_0$ and $\mathbf{b}(r_0) = -\mathrm{d}\mathbf{b}(r_0)/\mathrm{d}r_0$. Then the boundary condition (80) and Eq. (82) lead a system of singular integral equations:

$$\frac{k_{\mathrm{I}}}{2\pi}\int_b^c \frac{T_0}{r-r_0}\mathrm{d}r_0 + \frac{k_{\mathrm{I}}}{2\pi}\int_b^c K_t(r,r_0)T_0\mathrm{d}r_0 = h_0\cos(\omega) + h_\theta^{\mathrm{c}}(r,\omega), \tag{87}$$

where

$$\begin{aligned} K_t(r,r_0) =& -\frac{k_{\mathrm{I}}-k_{\mathrm{II}}}{k+k_{\mathrm{II}}}\mathrm{Re}\left[\frac{\mu}{r\mu - r_0\overline{\mu}}\right] + \frac{k_{\mathrm{II}}}{k_{\mathrm{I}}+k_{\mathrm{II}}}\mathrm{Re}\left[\frac{1}{r-r_0}\left(1-\sqrt{\frac{(r_0\mu-a)(r_0\mu-b)}{(r\mu-a)(r\mu-b)}}\right)\right. \\ &\left. + \frac{\mu}{r\mu-r_0\overline{\mu}}\left(1-\sqrt{\frac{(r_0\overline{\mu}-a)(r_0\overline{\mu}-b)}{(r\mu-a)(r\mu-b)}}\right) - \frac{2\mu}{\sqrt{(r\mu-a)(r\mu-b)}}\right], \\ h_\theta^{\mathrm{c}}(r,\omega) =& h_0\mathrm{Re}\left[\mu\left(1-\frac{r\mu-(a+b)/2}{\sqrt{(r\mu-a)(r\mu-b)}}\right)\right] \end{aligned} \tag{88}$$

and

$$\frac{1}{\pi}\int_b^c \frac{\mathscr{A}_b(\omega)}{r-r_0}\mathbf{b}\,\mathrm{d}r_0 + \frac{1}{\pi}\int_b^c K_b(r,r_0)\mathbf{b}\,\mathrm{d}r_0 + \frac{1}{2\pi}\int_b^c K_{bt}(r,r_0)T_0\,\mathrm{d}r_0 = \Omega_2 p_0 + \mathbf{t}_\theta^{c}(r,\omega) \tag{89}$$

in which

$$\begin{aligned} \mathscr{A}_b(\omega) =& \mathrm{Im}\left[\Omega_2 B_{\mathrm{I}} \ll \frac{1}{\zeta} \gg B_{\mathrm{I}}^{\mathrm{T}} + \Omega_1 B_{\mathrm{I}} \ll \frac{p_\alpha}{\zeta} \gg B_{\mathrm{I}}^{\mathrm{T}}\right] \\ K_b(r,r_0) =& \sum_{k=1}^{3}\mathrm{Im}\left[\Omega_2 B_{\mathrm{I}} \ll \frac{1}{r\zeta - r_0\overline{\zeta}_k} \gg B_{\mathrm{I}}^{-1}(I-2NL_{\mathrm{I}}^{-1})\overline{B}_{\mathrm{I}}I_k\overline{B}_{\mathrm{I}}^{\mathrm{T}}\right. \\ & + \Omega_1 B_{\mathrm{I}} \ll \frac{p_\alpha}{r\zeta - r_0\overline{\zeta}_k} \gg B_{\mathrm{I}}^{-1}(I-2NL_{\mathrm{I}}^{-1})\overline{B}_{\mathrm{I}}I_k\overline{B}_{\mathrm{I}}^{\mathrm{T}} \\ & \left. - \frac{2}{\pi}\Omega_2 N(v\mathbf{Y}_k(r\zeta, r_0\zeta_k;\epsilon)v^{-1}(N+\overline{N})^{-1}\mathscr{A}_k - v\mathbf{Y}_k(r\zeta, r_0\overline{\zeta}_k;\epsilon)v^{-1}(N+\overline{N})^{-1}\overline{\mathscr{A}}_k)\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{\pi}\Omega_1 N(v \ll p_\alpha \gg \mathbf{Y}_k(r\zeta, r_0\zeta_k;\epsilon)v^{-1}(N+\overline{N})^{-1}\mathscr{A}_k \\
&- v \ll p_\alpha \gg \mathbf{Y}_k(r\zeta, r_0\overline{\zeta}_k;\epsilon)v^{-1}(N+N)^{-1}\overline{\mathscr{A}}_k), \\
K_{bt}(r,r_0) = {}& \Omega_2 \mathrm{Im}[B_\mathrm{I} \ll \log(r\zeta - r_0\mu \gg B_\mathrm{I}^{-1}N(\overline{M}_\mathrm{II}D + \mathrm{i}C) \\
&+ D_\mathrm{I}(\log(r\mu - r_0\mu) - \frac{k_\mathrm{I}-k_\mathrm{II}}{k_\mathrm{I}+k_\mathrm{II}}\log(r\mu - r_0\overline{\mu})) + \overline{D}_\mathrm{int}(\widetilde{y}(r\mu, r_0\mu) + \widetilde{y}(r\mu, r_0\overline{\mu}))] \\
&+ \Omega_1 \mathrm{Im}[B_\mathrm{I} \ll p_\alpha \log(r\zeta - r_0\mu \gg B_\mathrm{I}^{-1}N(\overline{M}_\mathrm{II}D + \mathrm{i}C) \\
&+ D_\mathrm{I}(\log(r\mu - r_0\mu) - \frac{k_\mathrm{I}-k_\mathrm{II}}{k_\mathrm{I}+k_\mathrm{II}}\log(r\mu - r_0\overline{\mu}))\tau + \overline{D}_\mathrm{int}(\widetilde{y}(r\mu, r_0\mu) + \widetilde{y}(r\mu, r_0\overline{\mu}))\tau],
\end{aligned} \tag{90}
$$

where $\widetilde{I}_1 = \mathrm{diag}[0,1,1], \widetilde{I}_2 = \mathrm{diag}[1,0,1], \widetilde{I}_3 = \mathrm{diag}[1,1,0]$. Let

$$
r = \frac{(1+x)l}{2}, \quad r_0 = \frac{(1+t)l}{2}, \quad l = c - b, \tag{91}
$$

where, $|x| < 1$ and $|t| < 1$, then Eq. (87) and (89) may be rewritten as:

$$
\begin{aligned}
&\frac{k_\mathrm{I}}{2\pi}\int_{-1}^{1}\frac{T_0}{x-t}\mathrm{d}t + \frac{k_\mathrm{I}}{2\pi}\int_{-1}^{1}\widetilde{K}_t(x,t)T_0\,\mathrm{d}t = h_0\cos(\omega) + h_\theta^\mathrm{c}(x,\omega), \\
&\frac{1}{\pi}\int_{-1}^{1}\frac{\mathscr{A}_b(\omega)}{x-t}\mathbf{b}\,\mathrm{d}t + \frac{1}{\pi}\int_{-1}^{1}\widetilde{K}_b(x,t)\mathbf{b}\,\mathrm{d}t + \frac{1}{2\pi}\int_{-1}^{1}\widetilde{K}_{bt}(x,t)T_0\,\mathrm{d}t = \Omega_2 p_0 + \mathbf{t}_\theta^c(x,\omega),
\end{aligned} \tag{92}
$$

where $\widetilde{K}_t(x,t), \widetilde{K}_b(x,t)$ and $\widetilde{K}_{bt}(x,t)$ are obtained by substituting (91) in $K_t(r,r_0)$, $K_\mathrm{b}(r,r_0)$ and $K_{bt}(r,r_0)$, correspondingly. This system of singular equations involves two unknowns, namely T_0 and $\mathbf{b}$, which are coupled through the term $\widetilde{K}_{bt}$ in $(92)_2$. One can let (Erdogan et al., 1973):

$$
\begin{aligned}
T_0 &= w_1(t)\mathscr{T}(t), \quad w_1(t) = (1+t)^{-s_1}(1-t)^{\frac{1}{2}}, \\
\mathbf{b}(t) &= w_2(t)b(t), \quad w_2(t) = (1+t)^{-s_2}/(1-t)^{\frac{1}{2}}.
\end{aligned} \tag{93}
$$

Since the heat vortex density at both ends of the crack branched potion is bounded and the singularity at the intersection point of the main crack and the branched crack is of order less then $\frac{1}{2}$, then one can have $s_1 = -1/2$ and $s_2 = 1/2$ (Li and Kardomateas, 2005). Therefore, by using Gauss–Chebvshev integration Eq. $(92)_1$ can be solved. Once the solution for T_0 is obtained, substituting into $(92)_2$ and using Gauss–Jacobi integration formulas, the entire system of equations can be solved. Following a similar fashion as in Li and Kardomateas (2005), the numerical schemes for solving Eqs. $(92)_1$ and $(92)_2$ can be, respectively, written as

$$
\begin{aligned}
&\sum_{i=1}^{\mathrm{n}}\frac{1-t_i^2}{n+1}\mathscr{T}(t_i)\left[\frac{1}{t_i - x_k} - \widetilde{K}_t(t_i,x_k)\right] = \frac{2}{k_\mathrm{I}}[h_0\cos(\omega) + h_\theta^\mathrm{c}(x_k,\omega)], \\
&t_i = \cos\left(\frac{i\pi}{n+1}\right) \quad (i = 1,\ldots,n); \quad x_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{n+1}\right) \quad (k = 1,\ldots,n+1).
\end{aligned} \tag{94}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{1}{n}\left[\frac{\mathscr{A}_b(\omega)}{t_i - x_k} - \widetilde{K}_b(t_i,x_k)\right]b(t_i) = -\Omega_2 p_0 - \mathbf{t}_\theta^c(x_k,\omega) + \frac{1}{2\pi}\int_{-1}^{1}\widetilde{K}_{bt}(x_k,t)T_0\,\mathrm{d}t, \\
&\sum_{i}^{n}\frac{\pi}{n}b(t_i) = \Delta u, \\
&t_i = \cos\left(\pi\frac{2i-1}{2n}\right) \quad (i = 1,\ldots,n); \quad x_k = \cos\left(\frac{\pi k}{n}\right) \quad (i = 1,\ldots,n-1),
\end{aligned} \tag{95}
$$

where the second equation i.e. $(95)_2$ comes from the condition $\int_{-1}^{1} \mathbf{b}(t)\,dt = \Delta u$, which satisfies the continuity of displacement at the intersection point between the main crack and the branched portion. For an approximation, one may take $\int_{-1}^{1} \mathbf{b}(t)\,dt \approx 0$. But for more accurate computation, one would use Eq. (52) to evaluate the Δu by letting $a = -(L + l\cos(\omega))/2$, $b = (L + l\cos(\omega))/2$, and $x_1 = L/2$, where 'l' denotes the length of the branched portion of the crack and 'L' the length of the main crack. The integration of the third term on the right hand side of $(95)_1$ was performed by using Simpson's rule. Since the nodes used in (94) and (95) are different, the polynomial interpolations were also used to obtain the values of $\widetilde{K}_{bt}(x,t)$ and $T_0(t)$ from nodes in (94) for those values which are needed for the nodes in $(95)_1$.

The conventional stress intensity factors (SIFs) at the branched crack tip may be defined as

$$\mathbf{K} = [\mathbf{K}_{\mathrm{II}}, \mathbf{K}_{\mathrm{I}}, \mathbf{K}_{\mathrm{III}}]^{\mathrm{T}} = \lim_{r\to l^+} \sqrt{2\pi(r-l)}\,\mathbf{t}^{\mathrm{tot}}(r,\omega). \tag{96}$$

Using the technique given by Muskhelishvili (1953), the SIFs can be evaluated as

$$\begin{aligned}\mathbf{K} &= \lim_{r\to l^+} \sqrt{2\pi(r-l)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{\mathscr{A}_b(\omega)}{x-t}w_2(t)b(t)\mathrm{d}t + \frac{1}{2\pi}\int_{-1}^{1}\widetilde{K}_{bt}(x,t)w_1(t)\mathscr{T}(t)\mathrm{d}t\right]\\ &= \sqrt{\frac{\pi l}{2}}\mathscr{A}(\omega)\Omega_0(\omega)b(1),\end{aligned} \tag{97}$$

where an elementary relationship $\lim_{x\to 1^+}\sqrt{(x-1)}\log(x-1) \to 0$ is employed, and

$$\Omega_0(\omega) = \begin{vmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{98}$$

Once the onset of the branching of an interface crack happens, this crack usually propagates in one medium. Therefore, the energy release rate may be approximated as stated in Barnett and Asaro (1972) by

$$\mathscr{G}(\omega) = \frac{1}{2}\mathbf{K}^{\mathrm{T}}\widetilde{L}^{-1}\mathbf{K}, \quad \widetilde{L} = \Omega_0^{\mathrm{T}}(\omega)L\Omega_0(\omega), \tag{99}$$

where 'L' is the bi-material property matrix.

6. Numerical results

In this section, the influence of thermal loading on the delamination branching in composite bi-materials will be demonstrated. Two typical Graphite Epoxy composites were used as 'raw' or 'basic' material in the numerical simulation. The first material, called material-I was selected with thermo-elastic properties of: moduli in GPa: $E^{\mathrm{I}}_{11} = 5.69$, $E^{\mathrm{I}}_{22} = E^{\mathrm{I}}_{33} = 4.07$, $G^{\mathrm{I}}_{21} = 9.79$; Poisson's ratios: $v^{\mathrm{I}}_{21} = v^{\mathrm{I}}_{23} = v^{\mathrm{I}}_{31} = 0.01$; thermal conductivities in W/m/K: $k^{\mathrm{I}}_{11} = 42.1$, $k^{\mathrm{I}}_{22} = k^{\mathrm{I}}_{33} = 0.47$; thermal expansion coefficients in m/m/K: $\alpha^{\mathrm{I}}_{11} = 0.025 \times 10^{-6}$, $\alpha^{\mathrm{I}}_{22} = \alpha^{\mathrm{I}}_{33} = 32.4 \times 10^{-6}$. Thermo-elastic properties of the second raw material (material-II) read as: moduli in GPa: $E^{\mathrm{II}}_{11} = 2.312$, $E^{\mathrm{II}}_{22} = E^{\mathrm{II}}_{33} = 5.17$, $G^{\mathrm{II}}_{21} = 0.174$; Poisson's ratios: $v^{\mathrm{II}}_{21} = v^{\mathrm{II}}_{23} = v^{\mathrm{II}}_{31} = 0.1$; thermal conductivities in W/m/K: $k^{\mathrm{II}}_{11} = 53.7$, $k^{\mathrm{II}}_{22} = k^{\mathrm{II}}_{33} = 0.73$; thermal expansion coefficients in m/m/K: $\alpha^{\mathrm{II}}_{11} = 0.034 \times 10^{-6}$, $\alpha^{\mathrm{II}}_{22} = \alpha^{\mathrm{II}}_{33} = 34.2 \times 10^{-6}$. The angles θ_{I} and θ_{II} define the angles between material principal axis and the x_1 axis for upper and lower medium, respectively. The unit axial tension σ_{22} and unit heat flux q_0 in x_2 direction are considered to be the applied loading (Fig. 3).

Fig. 4 and 5 is the convergent illustration of the numerical scheme employed in Section 5. The bi-media used here consists of material-I as the upper medium and material-II as the lower medium and its bi-material parameter γ, defined in Eq. (39), equals 0.0662693. Depicted in Fig. 4 are the Mode I (K_{I}) and Mode II

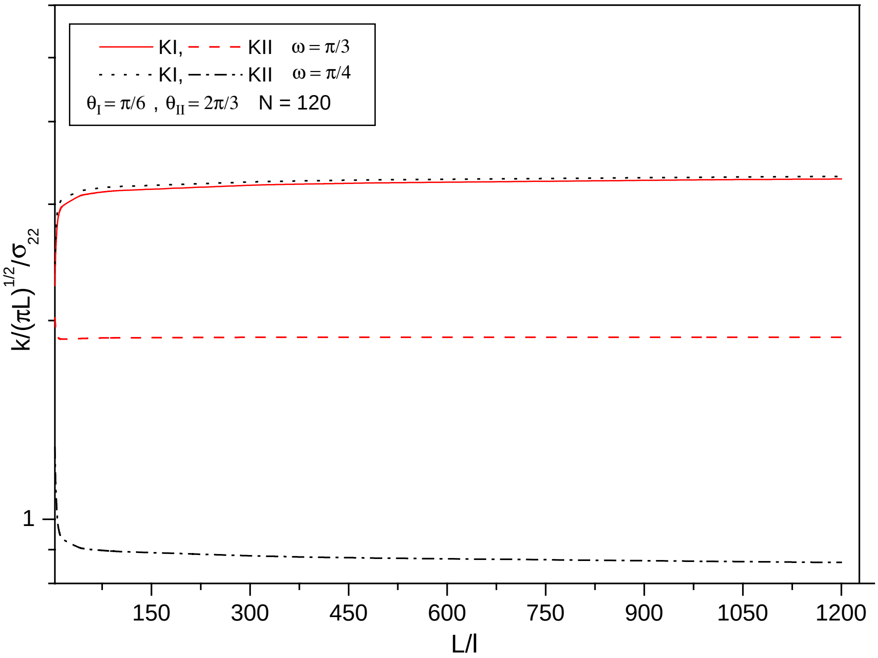

Fig. 4. Variations of stress intensity factors versus relative length (l/L) of branched crack.

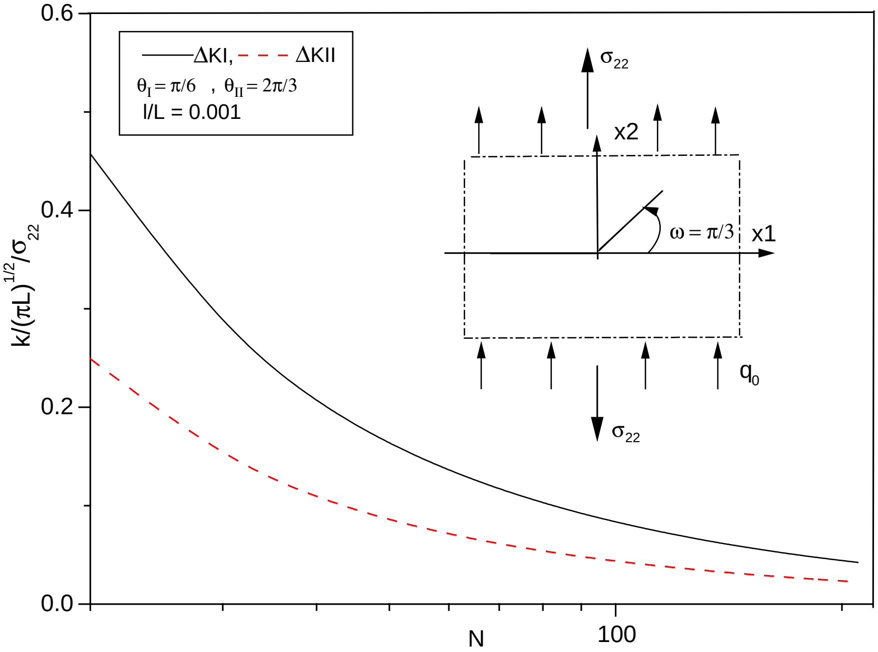

Fig. 5. Variations of stress intensity factors versus partition points of N.

(K_{II}) stress intensity factors around the branched crack tip as functions of L/l. The number of partition points in (94) and (95) is $n = 120$. Results of two cases were plotted, one for the assumed branching angle $\omega = \pi/3$ and the other for $\omega = \pi/4$. It can be seen that when $l/L > 0.1$, both values of K_I and K_{II} converge very well. When $l/L > 0.00125$, these values almost do not vary with the change of l/L. Therefore, the behavior of a branched crack with $l/L = 0.001$ can be considered as the behavior at the onset of interface crack branching. Usually, the onset of crack branching is of most interest in the study of interface crack problems. Fig. 5 gives the variation of K_I and K_{II} versus the change of partition points n. The value $l/L = 0.001$ was used here. To obtain these results, Δn was set to be 10 and ΔK is defined as the difference of the K evaluated at $n = i + 10$ and $n = i$ ($i > = 20$), respectively. It can be seen that $\Delta K \to 0$ as $n \to \infty$. This means K_I and K_{II} converge with increasing n. The plotting shows that one could get a good approximation by using $n = 60$ in the computation if one's computer memory is not big enough and the choice of partition points $n = 120$ in this paper would be very reasonable. Of course, if the computer memory permits, one can set n to be a big number. Thus, the infinitesimal crack branch was assumed to be $l/L = 0.001$ and the n was taken to be 120 in current paper.

6.1. Interface delamination branching for a general dissimilar anisotropic bi-media

As described in the above convergent study, the material properties (thermal and mechanical) of the upper and lower medium for this general bi-material structure are quite different. This type of bi-media can often be found in applications in many areas such as coating, electronic package, bio-mechanics structure, aerospace and nuclear power generator structure, etc. The components of a structure in these applications often have different thermal and mechanical properties and can operate under a severe temperature gradient. Therefore, the study of thermo-elastic interface crack branching propagation behavior is not only of theoretical importance but also of practical significance.

Fig. 6 and 7 show the mode I and mode II stress intensity factors and energy release rates versus the branching angle under different applied loading conditions. The orientation for the components of this bi-material media is $\theta_I = \pi/6$ and $\theta_{II} = -2\pi/3$. Three sets of results are plotted for three loading conditions: solid line for combined loading of unit σ_{22} (1 N/m^2) and q_0 (in W/m^2); dash-dot line for only unit σ_{22} applied; dash line for only unit q_0 applied. Several interesting observations can be made from the results in these two figures. In Fig. 6, the branching angle at which the K_I attains its maximum under combined loading is different from the corresponding angle under pure mechanical loading or thermal loading. For combined loading, $\omega = 51.44°$ and $K_{Imax} = 3.3394$, while for pure mechanical loading, $\omega = 43.45°$ and $K_{Imax} = 1.5507$ and for pure thermal loading, $\omega = 57.4665°$ and $K_{Imax} = 1.8198$. If the bi-material media originally under pure mechanical loading, then the K_{Imax} would increase by 115.3% due to additional thermal loading; or on the other hand, if the bi-material media originally under pure thermal loading, the $K_{I\text{-}max}$ *then would increase by 83.5% with the additional mechanical loading applied. The results for energy release rate* G are plotted in Fig. 7 and they share a similar tendency as those for K_I in Fig. 6. The angles at which the Gs reach their maximum values are also different: for combined loading, when $\omega = 40.94°$, $G_{max} = 9.7478$; for pure mechanical loading, $\omega = 34.78°$, $G_{max} = 2.3294$ and for pure thermal loading $\omega = 45.23°$, $G_{max} = 2.7123$. If one assumes that original loading is purely mechanical as in many engineering construction, then G_{max} would increase by 318.5% due to the additional thermal loading. One can see that although the energy release rate is a scalar value, its value under combined loading is not the summation of the values from the purely applied mechanical loading and purely thermal loading, in fact it is much bigger than the summation. The difference of these two values reflects the fact that a huge interaction energy would be produced once a heat flux added onto a mechanically loaded structure which includes defects. This observation can have significant implication in practical structure design. For example, according to the K-based criterion, interface cracks in a structure, usually operating in a constant temperature environment, would not grow from a sudden fire since the increased value of K may still fall into

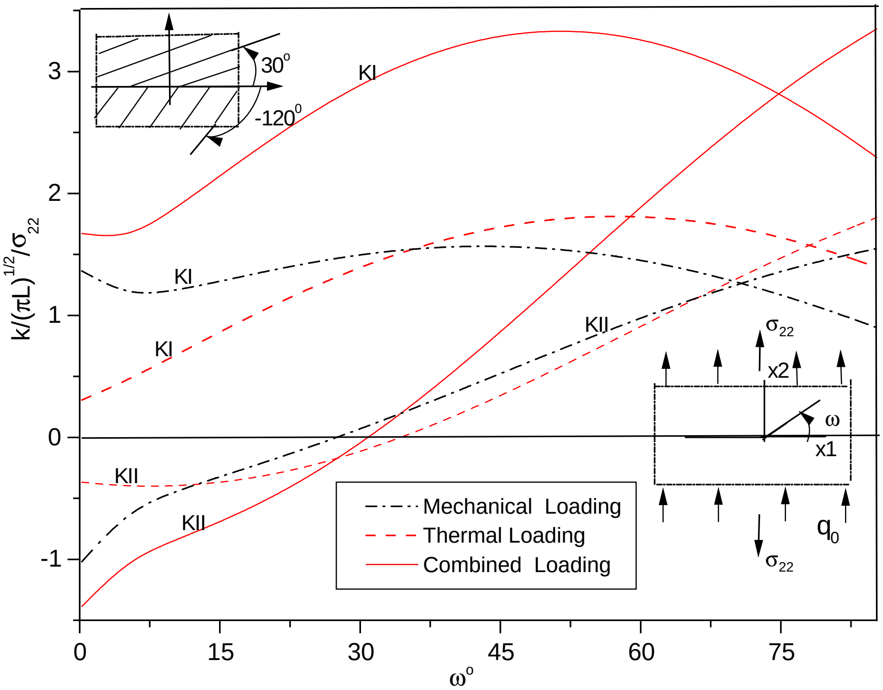

Fig. 6. Stress intensity factors for an anisotropic bi-medium ($\theta_{\mathrm{I}} = 30°$, $\theta_{\mathrm{II}} = -120^0$).

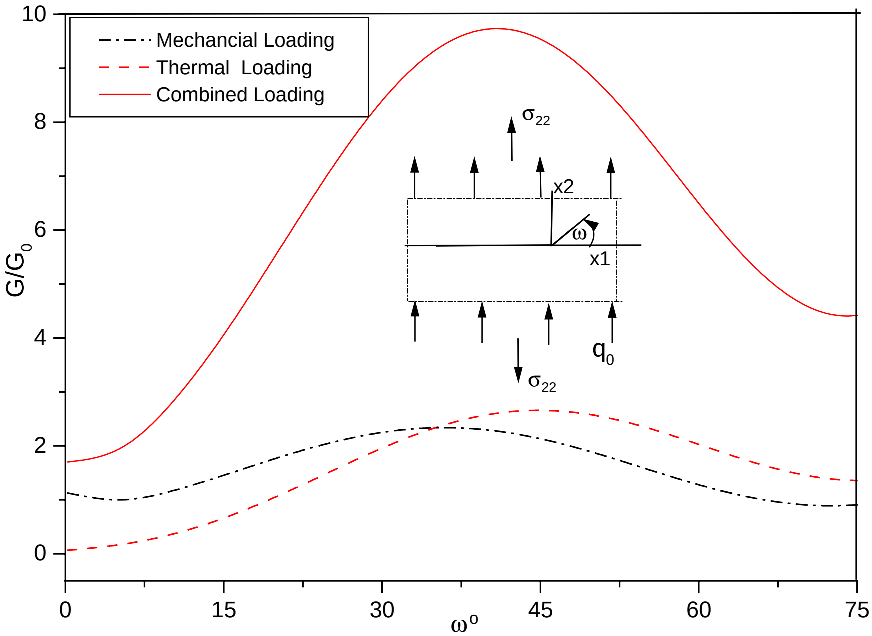

Fig. 7. Energy release rate for an anisotropic bi-medium ($\theta_{\mathrm{I}} = 30°$, $\theta_{\mathrm{II}} = -120^0$).

the design tolerance. However, there would be a strong interaction energy induced by the heat flux according to the energy release rate criterion, hence cracks in the structure may actually branch and grow quickly. Therefore, for the safety of the structure, a damage tolerance design should be based on a G-based criterion.

There are also some other interesting observations. In Fig. 6, one can see that when K_{I} reaches its maximum, the K_{II} does not equal to zero for each loading condition. This observation differs from that in monolithic isotropic medium or dissimilar quasi-bi-material media (which defined in next section) under pure mechanical loading, in which K_{I} is maximum when at the same time $K_{\mathrm{II}} = 0$. Two aspects may contribute to this difference: $\gamma \neq 0$ and/or the thermal loading effects. The above observations could suggest that the G-based criteria may be more suitable than the usual K-based criteria to predict thermo-elastic interface crack branching propagation for dissimilar anisotropic bi-material media.

6.2. Interface delamination branching in a quasi-bi-material media

For most dissimilar anisotropic bi-material media, their bi-material parameter γ usually is not zero. However, there is a set of bi-media whose constituents can be dissimilar but its bi-material parameter $\gamma = 0$. We define this type of bi-media as 'quasi-bi-material media'. Many engineering composites materials belong to this category. One way to produce such a composites is using one raw material and rotating the material axis with respect to the structure axis by different angles for the upper and lower components. It can be easily proven that the $\gamma = 0$ for this type of dissimilar bi-material media (Appendix D). Because of its special character the quasi-bi-material media is found to have some interesting behavior regarding the phenomenon of interface delamination branching.

Let us first consider a special loading condition case: pure mechanical loading [no thermal loading by setting $q_0 = 0.0$ in Eqs. (94) and (95)]. The 'basic' material elastic constants are similar to those in Miller and Stock (1989), i.e. moduli in GPa: $E_{11} = 4.89$ $E_{22} = E_{33} = 0.407$, $G_{21} = 0.731$; Poisson's ratios: $v_{21} = v_{23} = v_{31} = 0.02$. This raw material was used as upper medium. The lower medium was also made from this raw material but with the principal material axis being rotated $\theta_{\mathrm{II}} = -\pi/6$ with respect to the (x_1, x_2, x_3) coordinate system. The bi-material parameter γ equals zero, as proven in Appendix D.

The results of Mode I and Mode II stress intensity factors and energy release rate versus the branching angles are plotted in Figs. 8 and 9. Fig. 8 shows that the angles at which maximum values of K_{I} are attained ($\omega = 21.86°$ and $\omega = -11.83°$ for upper and lower medium, respectively) are the same angles where K_{II} approaches zero and there is a discontinuity in the stress intensity factors across the $\omega = 0°$ angle. These two observations are in good agreement with those in literature such as in Miller and Stock (1989), and this provides a kind of validation for the numerical scheme in the present paper. One remark: the materials used in this paper are similar to the ones used by Miller and Stock (1989), but not exactly the same; there are still some differences, such as the v_{ij} being different. Therefore, some disagreements in the comparison are expected.

One can easily see that in Fig. 9 the angles at which the maximum energy release rate occurs ($\omega = 10.9°$ and $\omega = -22.5°$ for upper and lower medium, respectively) are different from the angles for maximum K_{I}. It seems that there are two possible angles for the interface crack branching growth, one is $-11.83°$ which is based on maximum K_{I} (or zero K_{II}), the other one is $-22.5°$, which is based on maximum energy release rate. However, we often observe in experiments (La Saponara and Kardomateas, 2001) that crack branching usually tends to grow parallel to or along the fibers' orientation (which is 0° for the upper medium or $-30°$ for the lower medium in this case) in fiber-reinforced composite materials and this growth usually happens in weaker (more compliant) media as is always seen in sandwich debonding tests, i.e. debonding often branches into the core, almost never into the face sheet (La Saponara and Kardomateas, 2001). Here, the angle $\omega = -22.5°$ is very close to the orientation ($\theta_{\mathrm{II}} = -30°$) of the stiffer material axis of the weaker (or more compliant) component of this bi-material media. Therefore, these observations lead us to conclude

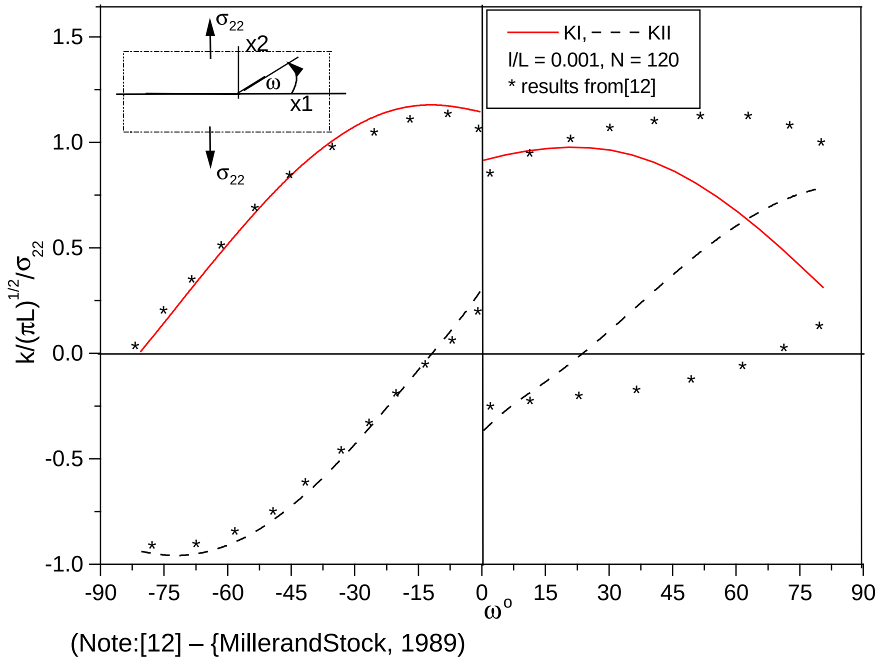

Fig. 8. SIFs at the branched crack tip vs. branching angle for a quasi-bi-material under pure tension loading.

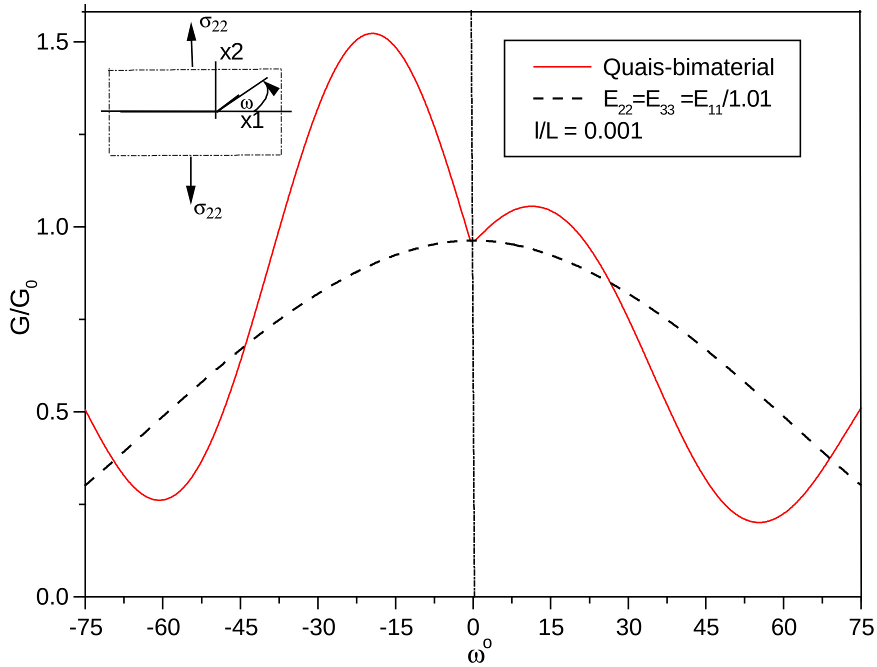

Fig. 9. Energy release rate for the branched crack vs. branching angle for a quasi-bi-material under pure tension loading.

that a maximum G-based criterion than a K-based criterion may be more accurate in predicting the interface crack branching for dissimilar anisotropic bi-material media. It should be noted at this point that the branching depends on both the max G_0 (the energy release rate if the interface crack) and the toughness of the interface/body. But the comparison made in this paper attempts to offer tentative guidelines that could help in the establishment of a correct failure criterion.

The influence of thermal constants' mismatches on the branching behavior of an interface delamination can be reflected by the difference between orientation angles θ_I and θ_{II}. The following example serves as such purpose. Figs. 10 and 11, respectively, show the results of Mode I and Mode II stress intensity factors and energy release rate versus the branching angles for three different bi-material media, which are formulated by letting $\theta_I = 0.0$ while $\theta_{II} = -\pi/6$, $\theta_{II} = -\pi/4$ and $\theta_{II} = -\pi/3$. Besides some observations similar to those in Figs. 6–9, several other observations can be made from Figs. 10 and 11. It can be seen that there is a discontinuity of the stress intensity factors and the energy release rate when the branching angle ω approaches $0^{\mp}$, respectively. This discontinuity for K_I and K_{II} was also shown on Fig. 8 and in the results of Miller and Stock (1989). But for pure mechanical loading there is no such discontinuity for the energy release rate as plotted in Fig. 9. This discontinuity on energy release rate in Fig. 11 shows another effect of thermal loading. Negative K_I (contact of the crack faces around the crack tip) (Li and Kardomateas, 2005) appears for the bi-material of $\theta_I = 0.0$, $\theta_{II} = -\pi/4$ when the branching angle $\omega > 13.75°$ or $-21.25° < \omega < 0°$ (the '−' sign means the interface delamination possibly branches into the lower medium), an observation being consist with the one in (Li and Kardomateas, 2005). Some other interesting results can also be observed in the plot of energy release rate. It can also be seen from Fig. 11 that the interface tends to branch into the lower medium, a result being consist with the observation in Fig. 7. But the corresponding maximum energy release rate, which is $G_{max} = 21.03$ for the bi-material media with $\theta_{II} = -\pi/6$, $G_{max} = 13.12$ for the bi-material media with $\theta_{II} = -\pi/4$, $G_{max} = 138.15$ for the bi-material media with

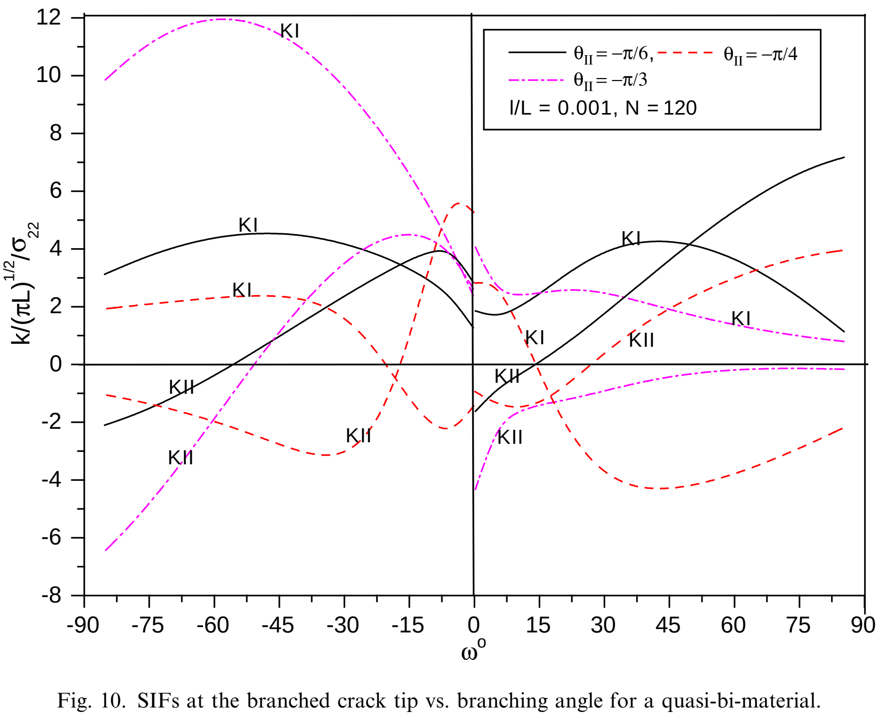

Fig. 10. SIFs at the branched crack tip vs. branching angle for a quasi-bi-material.

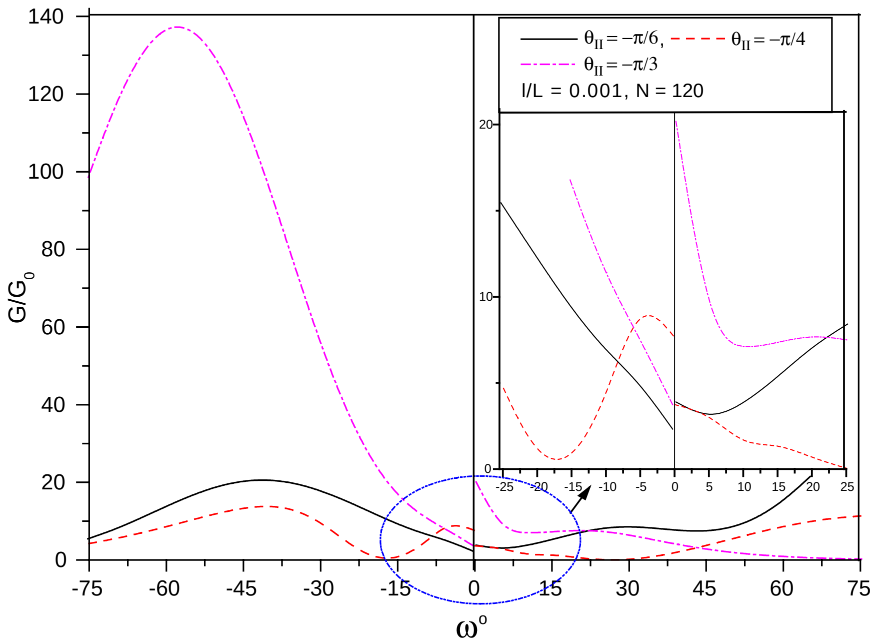

Fig. 11. Energy release rate for the branched crack vs. branching angle for a quasi-bi-material.

$\theta_{II} = -\pi/6$, does not simply increase as the orientation angle θ_{II} increases. In fact, G_{max} reaches its minimum value when $\theta_{II} = -\pi/4$. This observation may indicate that $\theta_{II} = -\pi/4$ could be the optimal orientation angle between the upper and lower medium for this bi-material media. Therefore, the results may be useful in optimal design for damage tolerance.

7. Conclusion

In this paper, a closed form solution is obtained for the thermo-elastic interaction between an interface crack and a dislocations (the thermal vortex and mechanical dislocation) in terms of matrix notation. The thermo-elastic interface crack/delamination branching phenomenon for dissimilar anisotropic bi-material media was subsequently investigated in detail. The influences of thermal loading on the onset of interface crack branching is addressed. The results of various cases are consistent with the observed fracture phenomena in composites and sandwich coupons with debonds. The observations in this study may suggest the following conclusions: (1). For general dissimilar anisotropic bi-material media, there usually exists a large interaction energy between the thermal loading and the mechanical loading for a structure with defects. This may have consequences, for example, in promoting failure when an imperfect bi-material structure is being exposed to a sudden fire; (2). G-based criterion may give more reasonable prediction than a K-based criterion for interface delamination branching angles of dissimilar anisotropic bi-media; (3). For some anisotropic bi-material media, negative K_I (overlapping of the delamination faces around the crack tip) is possible under certain loading conditions due to the thermal effects; (4). There exist an optimal orientation angle difference between the two constituents of a bi-material media. This optimal difference could minimize the value of maximum energy release rate. Therefore, the results in current work may also provide some useful guideline for damage tolerance engineering design.

Acknowledgements

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Appendix A. Summary of some basic thermo-anisotropic elasticity formulas

For a plane system, the non-trivial displacement $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ (with corresponding stress functions $\varphi = [\varphi_1, \varphi_2, \varphi_3]^{\mathrm{T}}$) and temperature distribution $T(x_1, x_2)$ which satisfy equations of equilibrium and heat conduction (with corresponding heat flux h_i, $i = 1,2$) are:

$$\begin{aligned} &\mathbf{u} = \mathbf{A}\phi(z_\alpha) + \overline{\mathbf{A}}\,\overline{\phi(z_\alpha)} + \mathbf{C}\chi(z_\tau) + \overline{\mathbf{C}}\,\overline{\chi(z_\tau)}; \quad \varphi = \mathbf{B}\phi(z_\alpha) + \overline{\mathbf{B}}\,\overline{\phi(z_\alpha)} + \mathbf{D}\chi(z_\tau) + \overline{\mathbf{D}}\,\overline{\chi(z_\tau)}, \\ &T(x_1,x_2) = \chi'(z_\tau) + \overline{\chi'(z_\tau)}; \quad \mathrm{h}_i = -(k_{i1} + \tau k_{i2})\chi''(z_\tau) - (k_{i1} + \overline{\tau} k_{i2})\overline{\chi''(z_\tau)}, \end{aligned} \tag{100}$$

where $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ are 3×3 matrices which satisfy the identity:

$$\begin{vmatrix} B^{\mathrm{T}} & A^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} & \overline{A}^{\mathrm{T}} \end{vmatrix} \times \begin{vmatrix} A & \overline{A} \\ B & \overline{B} \end{vmatrix} = \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix}; \tag{101}$$

$\mathbf{C}$ and $\mathbf{D}$ are 3×1 vectors; $\phi(z_\alpha)$ is a function vector and $\chi(z_\tau)$ is a scalar function; $z_\alpha = x_1 + p_\alpha x_2$ $(\alpha = 1,2,3)$ and $z_\tau = x_1 + \tau x_2$; the overbar $\overline{()}$ denotes the conjugate of a complex variable, the prime $'$ denotes differentiation with respect to z_α or z_τ; k_{i1}, k_{i2} $(i = 1,2)$ are coefficients of heat conductivity; the constant τ is the root with positive imaginary part of the equation

$$k_{22}\tau^2 + 2k_{12}\tau + k_{11} = 0; \tag{102}$$

the p_α, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constants which satisfy the following equations

$$N\begin{vmatrix} \mathbf{a} \\ \mathbf{b} \end{vmatrix} = p\begin{vmatrix} \mathbf{a} \\ \mathbf{b} \end{vmatrix}, \quad N = \begin{vmatrix} N_1 & N_2 \\ N_3 & N_1^{\mathrm{T}} \end{vmatrix}, \quad N\begin{vmatrix} \mathbf{c} \\ \mathbf{d} \end{vmatrix} = \tau\begin{vmatrix} \mathbf{c} \\ \mathbf{d} \end{vmatrix} - \begin{vmatrix} 0 & N_2 \\ \mathbf{I} & N_1^{\mathrm{T}} \end{vmatrix}\begin{vmatrix} \beta_1 \\ \beta_2 \end{vmatrix}, \quad (\beta_1)_i = \beta_{i1}, \quad (\beta_2)_i = \beta_{i2} \tag{103}$$

in which, $N_1 = -T^{-1}R^{\mathrm{T}}$, $N_2 = T^{-1}$, $\mathrm{N}_3 = RT^{-1}R^{\mathrm{T}} - Q$; the superscript '$T$' stands for the transpose of a matrix and

$$Q_{ik} = c_{i1k1}, \quad R_{ik} = c_{i1k2}, \quad T_{ik} = c_{i2k2}, \quad i,k = 1,2,3. \tag{104}$$

The function vector $\phi(z_\alpha)$ takes the form

$$\phi(z_\alpha) = \ll \mathbf{f}(z_\alpha) \gg \mathbf{q}; \quad \ll \mathbf{f}(z_\alpha) \gg = \mathrm{diag}[\mathbf{f}(z_1), \mathbf{f}(z_2), \mathbf{f}(z_3)], \tag{105}$$

where $\mathbf{f}(z_\alpha)$ and $\mathbf{q}$ are, respectively, the unknown functions and constants to be determined for a given problem and the $\ll \gg$ stands for a diagonal matrix. The stresses can be written in term of stress functions as:

$$\sigma_{i1} = -\frac{\partial\varphi_i}{\partial x_2} = -\varphi_{i,2}, \quad \sigma_{i2} = \frac{\partial\varphi_i}{\partial x_1} = \varphi_{i,1} = \varphi_i', \quad i = 1,2,3, \tag{106}$$

where the relationship $\frac{\partial\varphi_i}{\partial x_1} = \frac{\mathrm{d}\varphi_i}{\mathrm{d}z} = \varphi_i'$ is used in $(106)_2$.

If we let $k = k_{22}(\tau - \overline{\tau})/2\mathrm{i}$, then $k = \sqrt{k_{11}k_{22} - k_{12}^2}$ and

$$h_1 = \mathrm{i}k\tau\chi''(z_\tau) - \mathrm{i}k\overline{\tau\chi''(z_\tau)}, \quad h_2 = -\mathrm{i}k\chi''(z_\tau) + \mathrm{i}k\overline{\chi''(z_\tau)} \tag{107}$$

Here, three useful matrices are defined as

$$H = 2\mathrm{i}AA^{\mathrm{T}}, \quad L = -2\mathrm{i}BB^{\mathrm{T}}, \quad S = \mathrm{i}(2AB^{\mathrm{T}} - I), \tag{108}$$

where $I = \mathrm{diag}[1, 1, 1]$ is a unit matrix. It can be shown that H and L are symmetric and positive definite and SH, LS, $H^{-1}S$, S, SL^{-1} are anti-symmetric and the following relations can be

$$\begin{aligned} M &= -\mathrm{i}BA^{-1} = H^{-1}(I + \mathrm{i}S) = (I - \mathrm{i}S^{\mathrm{T}})H^{-1}, \\ M^{-1} &= \mathrm{i}AB^{-1} = L^{-1}(I + \mathrm{i}S^{\mathrm{T}}) = (I - \mathrm{i}S)L^{-1}. \end{aligned} \tag{109}$$

Appendix B. Contour integral for the interaction function

From Eqs. (34) and (44), the interaction stress functions read as

$$\Phi'(z) = \frac{1}{2\pi}X(z)\left[\int_a^b \frac{X_+^{-1}(x_1)}{x_1 - z}N^{-1}[p_0 + p_1^* x_1 + p_2^* \mathrm{i}\sqrt{(x_1 - a)(b - x_1)}]\,\mathrm{d}x_1 + Q_1(z)\right], \tag{110}$$

where

$$p_1^* = (\rho_1 + \rho_2)h_0^*, \quad p_2^* = (\rho_2 - \rho_1)h_0^*, \quad h_0^* = -\mathrm{i}\frac{k_{\mathrm{I}} + k_{\mathrm{II}}}{2k_{\mathrm{I}}k_{\mathrm{II}}}h_0. \tag{111}$$

By using contour integral one can get:

$$\begin{aligned}
J_1 &\equiv \frac{X(z)}{2\pi}\int_a^b \frac{X_+^{-1}(x_1)}{x_1 - z}N^{-1}p_0\,\mathrm{d}x_1 = v\{I - x(z)\Delta(z;\epsilon)[\Xi(z) + \Pi_1]\}v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_0); \\
J_2 &\equiv \frac{X(z)}{2\pi}\int_a^b \frac{x_1 X_+^{-1}(x_1)}{x_1 - z}N^{-1}p_1^*\,\mathrm{d}x_1 \\
&= v\{\Xi(z) - x(z)\Delta(z;\epsilon)[\Xi(z^2) + \Pi_1\Xi(z) - \Pi_2]\}v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_1^*); \\
J_3 &\equiv \frac{X(z)}{2\pi}\int_a^b \frac{\mathrm{i}\sqrt{(x_1 - a)(b - x_1)}X_+^{-1}(x_1)}{x_1 - z}N^{-1}p_2^*\,\mathrm{d}x_1 \\
&= v\{x^{-1}(z) - x(z)\Delta(z;\epsilon)[\Xi(z^2) - \Pi_3\Xi(z) + \Pi_4\}v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_2^*); \\
J_4 &\equiv \frac{X(z)}{2\pi}Q_1(z) = v\,\mathrm{diag}\left[0, 0, -\frac{(a-b)^2}{8}\frac{1}{\sqrt{(z-a)(z-b)}}\right]v^{-1}(N + \overline{N})^{-1}\mathrm{i}(p_1^* + p_2^*) \\
&\quad + v\,\mathrm{diag}\left[0, 0, \frac{1}{\sqrt{(z-a)(z-b)}}\right][\Pi_2 v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_1^*) + (\Pi_1^2 + \Pi_1\Pi_3 + \Pi_4)v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_2^*)] \\
&\quad + v\left[\frac{(a-b)^2}{8\sqrt{(z_1-a)(z_1-b)}}, \frac{(a-b)^2}{8\sqrt{(z_2-a)(z_2-b)}}, \frac{(a-b)^2}{8\sqrt{(z_3-a)(z_3-b)}}\right]v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_2^*) \\
&= v\,\mathrm{diag}\left[0, 0, \frac{(b-a)^2}{8\sqrt{(z-a)(z-b)}}\right]v^{-1}(N + \overline{N})^{-1}\mathrm{i}p_1^* \\
&\quad + v\,\mathrm{diag}\left[\frac{(b-a)^2}{8\sqrt{(z_1-a)(z_1-b)}}, \frac{(b-a)^2}{8\sqrt{(z_2-a)(z_2-b)}}, \frac{-(b-a)^2}{2\sqrt{(z_3-a)(z_3-b)}}\right]v^{-1}(N + \overline{N})^{-1}\mathrm{i}p_2^* \\
&= vx(z)\Pi_5 v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_1^*) + vx(z)\Pi_6 v^{-1}(N + \overline{N})^{-1}(\mathrm{i}p_2^*),
\end{aligned} \tag{112}$$

where

$$\begin{aligned}
&\Xi(z) = \mathrm{diag}[z_1, z_2, z_3],\\
&\Pi_1 = \mathrm{diag}\left[(b-a)\mathrm{i}\epsilon - \frac{a+b}{2}, (b-a)(-\mathrm{i}\epsilon) - \frac{a+b}{2}, -\frac{a+b}{2}\right],\\
&\Pi_2 = \mathrm{diag}\left[\left(\frac{b-a}{2}\right)^2(1+4\epsilon^2), \left(\frac{b-a}{2}\right)^2(1+4\epsilon^2), \left(\frac{b-a}{2}\right)^2\right],\\
&\Pi_3 = \mathrm{diag}[(a+b)+(b-a)\mathrm{i}\epsilon, (a+b)+(b-a)(-\mathrm{i}\epsilon), (a+b)],\\
&\Pi_4 = \mathrm{diag}\left[ab + \frac{b^2-a^2}{2}\mathrm{i}\epsilon - (1+4\epsilon^2)\left(\frac{b-a}{2}\right)^2, ab + \frac{b^2-a^2}{2}(-\mathrm{i}\epsilon) - (1+4\epsilon^2)\left(\frac{b-a}{2}\right)^2, ab - \left(\frac{b-a}{2}\right)^2\right],\\
&\Pi_5 = \mathrm{diag}\left[0, 0, \frac{(b-a)^2}{8}\right]; \quad \Pi_6 = \mathrm{diag}[1/8, 1/8, -1/2],
\end{aligned} \tag{113}$$

then

$$\Phi'(z) = J_1 + J_2 + J_3 + J_4. \tag{114}$$

Integration of Eq. (110) yields

$$\begin{aligned}
\Phi(z) = {} & v[\Xi(z) - x^{-1}(z)\Delta(z;\epsilon)]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_0) + v[\Xi(z^2) - x^{-1}\Delta(z;\epsilon)\Xi(z)]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_1^*)\\
& - v[x^{-1}\Delta(z;\epsilon)(\Xi(z) - \Pi_1 - \Pi_3]v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*) - vY_1(z;\epsilon)v^{-1}(N+\overline{N})^{-1}\mathrm{i}(p_1^* + p_2^*)\\
& - vY_2(z;\epsilon)[\Pi_2 v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_1^*) + (\Pi_1^2 + \Pi_1\Pi_3 + \Pi_4)v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*)]\\
& + vY_3(z)v^{-1}(N+\overline{N})^{-1}(\mathrm{i}p_2^*),
\end{aligned} \tag{115}$$

where

$$\begin{aligned}
Y_1(z;\epsilon) = {} & \mathrm{diag}\Bigg[\frac{(a-b)^{0.5+\mathrm{i}\epsilon}}{1.5-\mathrm{i}\epsilon}(z-a)^{1.5-\mathrm{i}\epsilon}{}_2F_1\left(1.5-\mathrm{i}\epsilon, -0.5-\mathrm{i}\epsilon, 2.5-\mathrm{i}\epsilon, \frac{z-a}{b-a}\right),\\
& \frac{(a-b)^{0.5-\mathrm{i}\epsilon}}{-1.5-\mathrm{i}\epsilon}(z-a)^{1.5+\mathrm{i}\epsilon}{}_2F_1\left(1.5+\mathrm{i}\epsilon, -0.5+\mathrm{i}\epsilon, 2.5+\mathrm{i}\epsilon, \frac{z-a}{b-a}\right),\\
& \sqrt{(z-a)(z-b)}(z-a+z-b)/4\Bigg],\\
Y_2(z;\epsilon) = {} & \mathrm{diag}\Bigg[\frac{(a-b)^{-0.5+\mathrm{i}\epsilon}}{0.5-\mathrm{i}\epsilon}(z-a)^{0.5-\mathrm{i}\epsilon}{}_2F_1\left(0.5-\mathrm{i}\epsilon, 0.5-\mathrm{i}\epsilon, 1.5-\mathrm{i}\epsilon, \frac{z-a}{b-a}\right),\\
& \frac{(a-b)^{-0.5-\mathrm{i}\epsilon}}{-0.5-\mathrm{i}\epsilon}(z-a)^{0.5+\mathrm{i}\epsilon}{}_2F_1\left(0.5+\mathrm{i}\epsilon, 0.5+\mathrm{i}\epsilon, 1.5+\mathrm{i}\epsilon, \frac{z-a}{b-a}\right), 0\Bigg],\\
Y_3(z_\alpha) = {} & \mathrm{diag}\Big[\sqrt{(z_1-a)(z_1-b)}(z_1-a+z_1-b)/4, \sqrt{(z_2-a)(z_2-b)}(z_2-a+z_2-b)/4,\\
& \sqrt{(z_3-a)(z_3-b)}(z_3-a+z_3-b)/4\Big],
\end{aligned} \tag{116}$$

in which, ${}_2F_1(\alpha_r;\gamma_s;z)$ is a generalized hypergeometric function with $\alpha_1 = 0.5 - \mathrm{i}\epsilon$, $\alpha_2 = 0.5 - \mathrm{i}\epsilon$, $\gamma_1 = 1.5 - \mathrm{i}\epsilon$, $z = \frac{z-a}{b-a}$ (Lebedev, 1972).

Appendix C. Solution to the thermal-dislocation of bi-media

From the boundary condition $(65)_{1,2}$ along the interface, one can obtain

$$\begin{aligned}&\mathrm{Re}\left[\frac{q_0\tau}{x_1-z_{\tau0}}+\frac{q_{1\tau}}{x_1-\bar{z}_{\tau0}}\right]=\mathrm{Re}\left[\frac{q_{2\tau}}{x_1-z_{\tau0}}\right],\\ &k_{\mathrm{I}}\mathrm{Im}\left[\frac{q_{0\tau}}{(x_1-z_{\tau0})^2}+\frac{q_{1\tau}}{(x_1-\bar{z}_{\tau0})^2}\right]=k_{\mathrm{II}}\mathrm{Im}\left[\frac{q_{2\tau}}{(x_1-z_{\tau0})^2}\right].\end{aligned}\tag{117}$$

Differentiation of $(117)_1$ with respect to x_1 gives

$$\mathrm{Re}\left[\frac{q_{0\tau}}{(x_1-z_{\tau0})^2}+\frac{q_{1\tau}}{(x_1-\bar{z}_{\tau0})^2}\right]=\mathrm{Re}\left[\frac{q_{2\tau}}{(x_1-z_{\tau0})^2}\right].\tag{118}$$

Solving Eqs. $(117)_2$ and (118) leads to

$$q_{1\tau}=\frac{k_{\mathrm{I}}-k_{\mathrm{II}}}{k_{\mathrm{I}}+k_{\mathrm{II}}}\bar{q}_{0\tau},\quad q_{2\tau}=\frac{2k_{\mathrm{I}}}{k_{\mathrm{I}}+k_{\mathrm{II}}}q_{0\tau}.\tag{119}$$

The boundary condition $(65)_{3,4}$ along the interface yields:

$$\begin{aligned}&\sum_1^3\{[A_{\mathrm{I}}\log(x_1-z_{d0k})I_kq_{d0}+\bar{A}_{\mathrm{I}}\log(x_1-\bar{z}_{d0k})I_k\bar{q}_{d0}]+[A_{\mathrm{I}}\log(x_1-\bar{z}_{d0k})q_{1k}+\bar{A}_{\mathrm{I}}\log(x_1-z_{d0k})\bar{q}_{1k}]\}\\
&\quad+[A_{\mathrm{I}}\log(x_1-z_{\tau0})q_{1d\tau}+\bar{A}_{\mathrm{I}}\log(x_1-\bar{z}_{\tau0})\bar{q}_{1d\tau}]+[C_{\mathrm{I}}\log(x_1-z_{\tau0})q_{0\tau}+\bar{C}_{\mathrm{I}}\log(x_1-\bar{z}_{\tau0})\bar{q}_{0\tau}]\\
&\quad+[C_{\mathrm{I}}\log(x_1-\bar{z}_{\tau0})q_{1\tau}+\bar{C}_{\mathrm{I}}\log(x_1-z_{\tau0})\bar{q}_{1\tau}]=\sum_1^3[A_{\mathrm{II}}\log(x_1-z_{d0k})q_{2k}+\bar{A}_{\mathrm{II}}\log(x_1-\bar{z}_{d0k})\bar{q}_{2k}]\\
&\quad+[A_{\mathrm{II}}\log(x_1-\bar{z}_{\tau0})q_{2d\tau}+\bar{A}_{\mathrm{II}}\log(x_1-z_{\tau0})\bar{q}_{2d\tau}]+[C_{\mathrm{II}}\log(x_1-z_{\tau0})q_{2\tau}+\bar{C}_{\mathrm{II}}\log(x_1-\bar{z}_{\tau0})\bar{q}_2\tau];\\
&\sum_1^3\{[B_{\mathrm{I}}\log(x_1-z_{d0k})I_kq_{d0}+\bar{B}_{\mathrm{I}}\log(x_1-\bar{z}_{d0k})I_k\bar{q}_{d0}]+[B_{\mathrm{I}}\log(x_1-\bar{z}_{d0k})q_{1k}+\bar{B}_{\mathrm{I}}\log(x_1-z_{d0k})\bar{q}_{1k}]\}\\
&\quad+[B_{\mathrm{I}}\log(x_1-z_{\tau0})q_{1d\tau}+\bar{B}_{\mathrm{I}}\log(x_1-\bar{z}_{\tau0})\bar{q}_{1d\tau}]+[D_{\mathrm{I}}\log(x_1-z_{\tau0})q_{0\tau}+\bar{D}_{\mathrm{I}}\log(x_1-\bar{z}_{\tau0})\bar{q}_{0\tau}]\\
&\quad+[D_{\mathrm{I}}\log(x_1-\bar{z}_{\tau0})q_{1\tau}+\bar{D}_{\mathrm{I}}\log(x_1-z_{\tau0})\bar{q}_{1\tau}]=\sum_1^3[B_{\mathrm{II}}\log(x_1-z_{d0k})q_{2k}+\bar{B}_{\mathrm{II}}\log(x_1-\bar{z}_{d0k})\bar{q}_{2k}]\\
&\quad+[B_{\mathrm{II}}\log(x_1-\bar{z}_{\tau0})q_{2d\tau}+\bar{B}_{\mathrm{II}}\log(x_1-z_{\tau0})\bar{q}_{2d\tau}]+[D_{\mathrm{II}}\log(x_1-z_{\tau0})q_{2\tau}+\bar{D}_{\mathrm{II}}\log(x_1-\bar{z}_{\tau0})\bar{q}_{2\tau}].\end{aligned}\tag{120}$$

Following two sets of equations can be derived by grouping the coefficients of terms $\log(x_1-z_{d0\mathrm{k}})$, and $\log(x_1-z_{\tau0})$ in the above equation:

$$\begin{aligned}&-\bar{A}_{\mathrm{I}}\bar{q}_{1k}+A_{\mathrm{II}}q_{2k}=A_{\mathrm{I}}I_kq_{d0},\\ &-\bar{B}_{\mathrm{I}}\bar{q}_{1k}+B_{\mathrm{II}}q_{2k}=B_{\mathrm{I}}I_kq_{d0}\end{aligned}\tag{121}$$

and

$$\begin{aligned}&A_{\mathrm{I}}q_{1d\tau}-\bar{A}_{\mathrm{II}}\bar{q}_{2d\tau}=C_{\mathrm{II}}q_{2\tau}-\bar{C}_{\mathrm{I}}\bar{q}_{1\tau}-C_{\mathrm{I}}q_{0\tau},\\ &B_{\mathrm{I}}q_{1d\tau}-\bar{B}_{\mathrm{II}}\bar{q}_{2d\tau}=D_{\mathrm{II}}q_{2\tau}-\bar{D}_{\mathrm{I}}\bar{q}_{1\tau}-D_{\mathrm{I}}q_{0\tau}.\end{aligned}\tag{122}$$

Eqs. (121) and (122), respectively, give

$$B_{\mathrm{I}}q_{1k}=N[-N^{-1}+2L_1^{-1}]\bar{B}_{\mathrm{I}}I_k\bar{q}_{d0},\quad B_{\mathrm{II}}q_{2k}=2\bar{N}L_1^{-1}B_{\mathrm{I}}I_kq_{d0}\tag{123}$$

and

$$B_{\rm I}q_{1d\tau} = N[\overline{M}_{\rm II}^{-1}D + {\rm i}C]q_{0\tau}, \quad B_{\rm II}q_{2d\tau} = -\overline{N}[\overline{M}_{\rm I}^{-1}\overline{D} + {\rm i}\overline{C}]q_{0\tau}, \tag{124}$$

where

$$C = \frac{2k_{\rm I}}{k_{\rm I}+k_{\rm II}}C_{\rm II} - \frac{k_{\rm I}-k_{\rm II}}{k_{\rm I}+k_{\rm II}}\overline{C}_{\rm I} - C_{\rm I}, \quad D = \frac{2k_{\rm I}}{k_{\rm I}+k_{\rm II}}D_{\rm II} - \frac{k_{\rm I}-k_{\rm II}}{k_{\rm I}+k_{\rm II}}\overline{D}_{\rm I} - D_{\rm I}. \tag{125}$$

Appendix D. Proof $\gamma = 0$ for ”quasi-bi-materials” (same “basic” material but different fiber orientations for the two phases)

It is easily to show SL^{-1} is antisymmetric. Actually, from the definition of matrices S, L and using Eq. (109)

$$SL^{-1} = {\rm i}(2AB^{\rm T} - I)(-2{\rm i}BB^{\rm T})^{-1} = \frac{B^{-{\rm T}}B^{-1}}{2} - AB^{-1} = \frac{B^{-{\rm T}}B^{-1}}{2} - L^{-1}(S^{\rm T} - {\rm i}I) = -L^{-1}S^{\rm T}$$
$$= -[SL^{-1}]^{\rm T}. \tag{126}$$

It follows that $W = S_1L_1 - S_2L_2$ is antisymmetric.

If x_3 is an axis of material symmetry, then the third components of the first and second vector in matrix A and B are zero, so are the first and second component of the third vector. Therefore, the matrix SL^{-1} can only has the following form

$$SL^{-1} = \begin{vmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & d \end{vmatrix}. \tag{127}$$

Hence,

$$S_2L_2^{-1} = \Omega^{\rm T}S_1\Omega[\Omega^{\rm T}L_1^{-1}\Omega]^{-1} = \Omega^{\rm T}S_1L_1^{-1}\Omega = \begin{vmatrix} 0 & b[\cos(\omega)^2 + \sin(\omega)^2] & 0 \\ -b[\cos(\omega)^2 + \sin(\omega)^2] & 0 & 0 \\ 0 & 0 & d \end{vmatrix} = S_1L_1^{-1}. \tag{128}$$

This shows that W is a null matrix, then it follows that the bi-material parameter $\epsilon = 0.0$ by definition of W.

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